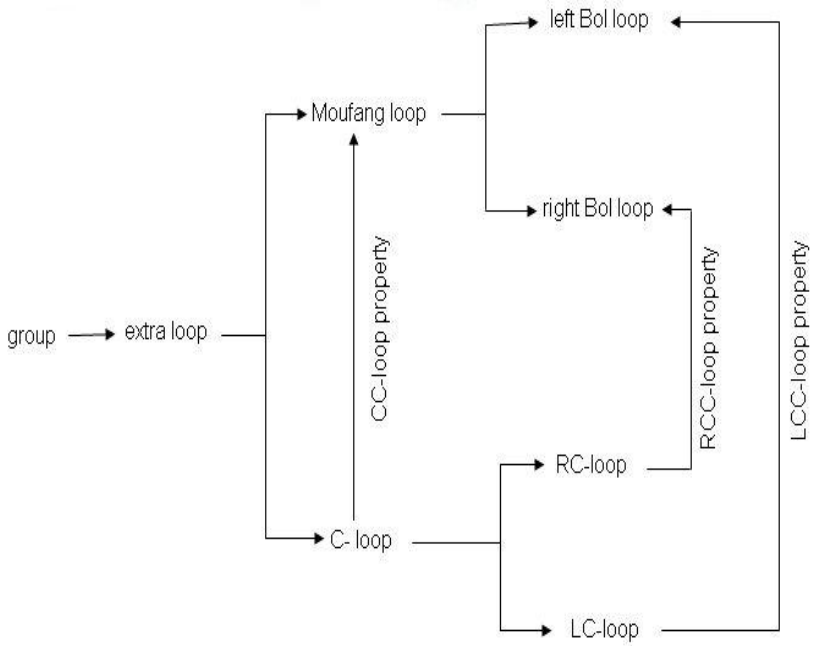
JAIYEOLA TEMITOPE GBOLAHAN

A STUDY OF NEW CONCEPTS IN SMARANDACHE QUASIGROUPS AND LOOPS



ILQ

2009

A Study Of New Concepts In Smarandache Quasigroups And Loops

Jaiyeola Temitope Gbolahan

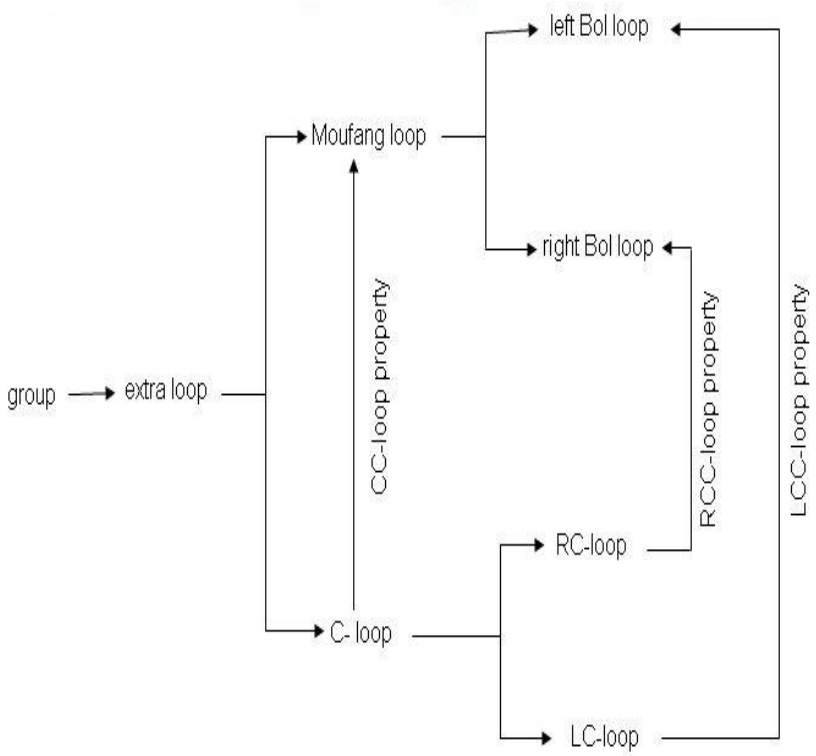
Department of Mathematics

Faculty of Science

Obafemi Awolowo University

Ile-Ife, 220005, Nigeria

jaiyeolatemitope@yahoo.com, tjayeola@oauife.edu.ng



ILQ

2009

This **book can be ordered** in a paper bound reprint from:

Books on Demand
ProQuest Information & Learning
(University of Microfilm International)
300 N. Zeeb Road
P.O. Box 1346, Ann Arbor
MI 48106-1346, USA

Copyright 2009 by InfoLearnQuest (Ann Arbor) and Temitope Gbolahan Jaiyeola

Many books can be downloaded from the following
Digital Library of Sciences:
http://fs.gallup.unm.edu/ /eBooks-otherformats.htm

Peer Reviewers:

Dr. Sukanto Bhattacharya, Department of Business Administration, Alaska Pacific University, U.S.A.

Prof. Mihaly Bencze, Department of Mathematics, Áprily Lajos College, Braşov, Romania.

Drd. Marian Popescu and Drd. FlorentinPopescu, Craiova, Romania.

(ISBN-10): 1-59973-083-9
(ISBN-13): 978-1-59973-083-7
(EAN): 9781599730837

Printed in the United States of America

PREFACE

The various areas where loop theory originated and through which it moved during the early part of its 70 years of history can be mapped and fitted not only in a geographical and a chronological sense but also conceptually. Loop theory is of course a relatively young subject which continues to grow day by day. To give an example, when somebody asks. 'What is a loop?', the simplest way to explain is to say, 'it is a group without associativity'. This is true, but it is not the whole truth. It is essential to emphasize that loop theory is not just a generalization of group theory but a discipline of its own, originating from and still moving within four basic research areas ; algebra, geometry, topology and combinatorics. Tremendous contributions to the theory of loops can be traced mainly back to Europe at large and America. Asia and Africa are the only two continents that seem left out of the history of loop theory. But in 2002, the Asian axis surfaced in the history of loop theory when W. B. Vasantha Kandasamy, an Indian female Algebraist initiated the study of Smarandache loops(i.e. loops with at least a non-trivial subgroup) in her book titled "Smarandache Loops" and her paper titled "Smarandache Loops" published by various publishing houses. She has also published some books and research articles on related subjects like Smarandache Groupoids, Smarandache Semigroups, Smarandache Semirings, Smarandache Semifields, Smarandache Semivector Spaces, Smarandache Linear Algebra, Smarandache Bialgebraic Structures and Smarandache N-Algebraic Structures as a sole author and co-authored with the prolific and dynamic scientist, Florentin Smarandache.

In 2004, when I was at the University of Agriculture Abeokuta(UNAAB), in Nigeria for my Masters Degree in Mathematics(I actually started studying quasigroups and loop theory then) under the supervision of Dr. J. O. Adeniran, Prof. A. R. T. Solarin informed us that a student of his in Tanzania was working on a new class of loop called Smarandache loop. We were able to lay our hands on the book and paper of Vasantha Kandasamy on Smarandache

loops and at that point caught a vision for this class of loop. I believed it can be turned into a theory of its own just like quasigroup and loop theory has. The first part of the vision was to introduce some new Smarandache notions in quasigroups and loops which are entirely different from the over 75 Smarandache concepts on loops. The second part of the vision later on came in 2006 when I was at Obafemi Awolowo University(OAU), in Nigeria. Students were on rampage and the university was closed down. I later on got the information that the violent acts were actually perpetrated by some set of students including the executives of the students' union. The situation can be algebraically modelled as a set(the set of all students in OAU) with a non-trivial subset(the set of unscrupulous elements that were the master minders of the violent acts) such that the set with a binary operation is expected to obey some set of axioms(the regulation of the OAU authority) but which the elements(students) in the subset do not obey at all or do obey but at the same time perpetrate some other acts(axioms or laws or equations) that are not in the OAU university regulations. To me, this exactly fits into the definition of a Smarandache structure on a set according to Raul Padilla of 1998. This is consider as "The study of both the good and the bad". In the past, a model either studies and investigate a bad situation and circumstance or a good situation.

This book is a compilation of results on some new Smarandache concepts in Smarandache; groupoids, quasigroups and loops which I have so far published in the "Scientia Magna Journal" and the "International Journal of Mathematical Combinatorics" both in China, and pin points the inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed. It is more of a monograph. A pre-requisite to the readability of this monograph is a basic knowledge of the theory of quasigroups and loops. Any graduate student or researcher who has a good understanding of the some popular books in the theory of quasigroups and loops like "R. H. Bruck, A Survey of Binary Systems, Springer Verlag, 1958", "Orin Chein, H. O. Pflugfelder and J. D. H. Smith, Quasigroups and Loops: Theory and Applications, Heldermann Verlag, 1990",

"H. O. Pflugfelder, Quasigroups and Loops: Introduction, Heldermann Verlag, 1990" will be able to comprehend the content of this monograph.

This monograph is structured into six chapters. The first chapter is an introduction to the theory quasigroups and loops with much attention paid to those quasigroup and loop concepts whose Smarandache versions are to be studied in the other chapters. In chapter two, the holomorphic structures of Smarandache loops of Bol-Moufang type and Smarandache loops of non-Bol-Moufang type are studied. The holomorphic structure of Smarandache cross inverse property quasigroups is found useful and applicable for double cryptography and this demonstrated with what is called the Smarandache Keedwell cross inverse property quasigroup. In the third chapter, the notion of parastrophy is introduced into Smarandache quasigroups and studied. Chapter four studies the universality of some Smarandache loops of Bol-Moufang type. In chapter five, the notion of Smarandache isotopism is introduced and studied in Smarandache quasigroups and loops and the cardinality of the Smarandache isomorphism classes of some Smarandache loops of small orders are investigated. In chapter six, by introducing Smarandache special mappings in Smarandache groupoids, the Smarandache Bryant-Schneider group of a Smaranache loop is developed and studied via the notion of Smarandache isotopism of chapter five. Some cardinality formulas are also gotten for the Smarandache Bryant-Schneider group of a Smarandache loop.

I so much appreciate the words of encouragement of Dr. Minh Perez of the American Research Press, U.S.A. and the Editor in Chief of the Smarandache Notions Journal during the years of the development of most of the results that are proved in this monograph. He has been a wonderful friend of mine who took great interest in my academic career and challenged me to put these results together. I appreciate you for this.

Jaiyeola T. G.

January, 2009

Contents

Preface Page **iii**

Tables of Contents **vi**

List of tables **ix**

List of figures **x**

1 THEORY OF QUASIGROUPS AND LOOPS **1**

1.1 Groupoids, Quasigroups And Loops 1

1.1.1 Some Important Subloops Of Loops 3

1.1.2 Inner Mappings Of A Loop 5

1.1.3 Basic Quasigroups And Loops Properties 6

1.2 Varieties of Quasigroups And Loops 11

1.2.1 Quasigroups And Loops Of Bol-Moufang Types 11

1.3 Universality Of Loops 14

1.3.1 Universality Of Bol-Moufang Type Loops 17

1.4 Autotopisms Of A Loop 18

1.5 G-Loops 18

1.5.1 Special Mappings Of A Loop 19

1.6 Bryant-Schneider Group Of A Loop 20

1.7 Smarandache Quasigroups And Loops 21

2 HOLOMORPHY OF SMARANDACHE QUASIGROUPS AND LOOPS 24

2.1 Holomorph And Smarandache Holomorph 24

2.2 Smarandache Isotopism And Autotopism 27

2.3 Holomorphy Of Some Smarandache Loops 31

2.3.1 Holomorphy Of Smarandache Inverse Property Loops 35

2.3.2 Holomorphy Of Smarandache Conjugacy Closed Loops 40

2.3.3 Holomorphy Of Smarandache: Bol Loops, Central Loops, Extra Loops And Burn Loops 42

2.3.4 Holomorphy Of Smarandache: A-Loops, Homogeneous Loops 44

2.3.5 Holomorphy Of Smarandache: K-Loops, Bruck-Loops and Kikkawa-Loops 45

2.4 Double Cryptography Using The Smarandache Keedwell CIQ 46

3 PARASTROPHY OF SMARANDACHE QUASIGROUPS AND LOOPS 51

3.1 Parastrophy Of Quasigroups And Loops 51

3.2 Parastrophy Of Smarandache Quasigroups 55

4 UNIVERSALITY OF SMARANDACHE LOOPS OF BOL-MOUFANG TYPE 61

4.1 Smarandache f, g-Principal Isotopes 61

4.2 Universality Of Smarandache Loops 63

4.3 Universality of Smarandache Bol Loops, Moufang Loops and Extra Loops . . 65

4.4 Universality Of Smarandache Inverse Property Loops 74

5 SMARANDACHE ISOTOPY OF SMARANDACHE QUASIGROUPS AND LOOPS 79

5.1 Smarandache Isotopy And Isomorphy Classes 80
5.2 Smarandache f, g-Isotopes Of Smarandache Loops 87
5.3 G-Smarandache Loops 90

6 SMARANDACHE BRYANT SCHNEIDER GROUP OF SMARANDACHE LOOPS 92

6.1 Smarandache Special Maps 92
6.2 The Smarandache Bryant Schneider Group 93
6.3 The SBS Group Of A Smarandache f, g-Principal Isotope 95
6.4 Cardinality Formulas 101

Bibliography 111

Index 120

List of Tables

2.1 A Smarandache Quasigroup 29

2.2 A Smarandache Quasigroup Isotope 29

2.3 A Smarandache Semigroup 30

2.4 A Smarandache Semigroup 30

5.1 Enumeration of Isomorphy and Isotopy classes of finite loops of small order . 84

List of Figures

1.1 Varieties of Inverse Property Loops 10
1.2 Varieties of Loops of Bol-Moufang type 15

Chapter 1

THEORY OF QUASIGROUPS AND LOOPS

1.1 Groupoids, Quasigroups And Loops

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G.

If $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a groupoid or Magma.

If the system of equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions in G for x and y respectively, then $(G, \cdot)$ is called a quasigroup.

A quasigroup is therefore an algebra having a binary multiplication $x \cdot y$ usually written xy which satisfies the conditions that for any a, b in the quasigroup the equations

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y lying in the quasigroup.

If there exists a unique element $e \in G$ called the identity element such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a loop.

Let x be a fixed element in a groupoid $(G, \cdot)$. The so called left and right translation maps of G, L_x and R_x respectively can be defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations.

Definition 1.1.1 *A set* Π *of permutations on a set* G *is the representation of a loop* $(G, \cdot)$ *if and only if*

(i) $I \in \Pi$ *(identity mapping),*

(ii) Π *is transitive on* G*(i.e for all* $x, y \in G$*, there exists a unique* $\pi \in \Pi$ *such that* $x\pi = y$*),*

(iii) *if* $\alpha, \beta \in \Pi$ *and* $\alpha\beta^{-1}$ *fixes one element of* G*, then* $\alpha = \beta$*.*

The left and right representation of a loop G is denoted by

$$\Pi_\lambda(G, \cdot) = \Pi_\lambda(G) \qquad \text{and} \qquad \Pi_\rho(G, \cdot) = \Pi_\rho(G) \text{ respectively}$$

Since the left and right translation mappings of a loop are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} \qquad \text{and} \qquad x/y = xR_y^{-1}$$

and note that

$$x\backslash y = z \iff x \cdot z = y \qquad \text{and} \qquad x/y = z \iff z \cdot y = x.$$

Hence, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be stated as follows.

Definition 1.1.2 *A loop $(G, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that*

(i) $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ *for all* $x, y \in G$,

(ii) $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ *for all* $x, y \in G$ *and*

(iii) $x\backslash x = y/y$ *or* $e \cdot x = x$ *and* $x \cdot e = x$ *for all* $x, y \in G$.

It must be stipulated that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x \cdot (y\backslash z)$ respectively.

The basic text books on quasigroup, loops are Pflugfelder [72], Bruck [21], Chein, Pflugfelder and Smith [25], Dene and Keedwell [29], Goodaire, Jespers and Milies [37] and Vasantha Kandasamy [86].

1.1.1 Some Important Subloops Of Loops

Let $(L, \cdot)$ be a loop.

The left nucleus of L denoted by

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall \ x, y \in L\}.$$

The right nucleus of L denoted by

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall \ x, y \in L\}.$$

The middle nucleus of L denoted by

$$N_\mu(L,\cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall \ x, y \in L\}.$$

The nucleus of L denoted by

$$N(L,\cdot) = N_\lambda(L,\cdot) \cap N_\rho(L,\cdot) \cap N_\mu(L,\cdot).$$

The centrum of L denoted by

$$C(L,\cdot) = \{a \in L : ax = xa \ \forall \ x \in L\}.$$

The center of L denoted by

$$Z(L,\cdot) = N(L,\cdot) \cap C(L,\cdot).$$

Let a, b and c be three elements of a loop L. The loop commutator of a and b is the unique element (a, b) of L which satisfies

$$ab = (ba)(a, b)$$

and the loop associator of a, b and c is the unique element (a, b, c) of L which satisfies

$$(ab)c = \{a(bc)\}(a, b, c).$$

If X, Y, and Z are subsets of a loop L, we denote by (X, Y) and (X, Y, Z), respectively, the set of all commutators of the form (x, y) and all the associators of the form (x, y, z), where $x \in X, y \in Y, z \in Z$.

The symmetric group of $(L, \cdot)$ denoted by

$$SYM(L, \cdot) = \{U : L \to L \mid U \text{ is a permutation or a bijection}\}.$$

The set

$$\mathcal{M}\text{ult}_{\lambda}(L, \cdot) = \left\langle \{L_x, L_x^{-1} \ : \ x \in L\} \right\rangle$$

is called the left multiplication group of $(L, \cdot)$. $\mathcal{M}\text{ult}_{\lambda}(L, \cdot) \leq SYM(L, \cdot)$. The set

$$\mathcal{M}\text{ult}_{\rho}(L, \cdot) = \left\langle \{R_x, R_x^{-1} \ : \ x \in L\} \right\rangle$$

is called the right multiplication group of $(L, \cdot)$. $\mathcal{M}\text{ult}_{\rho}(L, \cdot) \leq SYM(L, \cdot)$. The set

$$\mathcal{M}\text{ult}(L, \cdot) = \left\langle \{R_x, R_x^{-1}, L_x, L_x^{-1} \ : \ x \in L\} \right\rangle$$

is called the multiplication group of $(L, \cdot)$. $\mathcal{M}\text{ult}(L, \cdot) \leq SYM(L, \cdot)$.

1.1.2 Inner Mappings Of A Loop

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}\text{ult}(G)$, then α is called an inner mapping and they form a group $\text{Inn}(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R(x, y) = R_x R_y R_{xy}^{-1}, \ L(x, y) = L_x L_y L_{yx}^{-1} \text{ and } T(x) = R_x L_x^{-1}$$

respectively form the left inner mapping group $\text{Inn}_{\lambda}(G)$, right inner mapping group $\text{Inn}_{\rho}(G)$ and the middle inner mapping $\text{Inn}_{\mu}(G)$. If

$$\text{Inn}_{\lambda}(G) \leq AUM(G), \ \text{Inn}_{\rho}(G) \leq AUM(G), \ \text{Inn}_{\mu}(G) \leq AUM(G) \text{ and } \text{Inn}(G) \leq AUM(G),$$

where $AUM(G)$ denotes the automorphism group of G, then G is called a left A-loop(A_λ-loop), right A-loop(A_ρ-loop), middle A-loop(A_μ-loop) and A-loop respectively. The definitions above are in accordance with those in [36].

1.1.3 Basic Quasigroups And Loops Properties

For associative binary systems, the concept of an inverse element is only meaningful if the system has an identity element. For example, in a group $(G, \cdot)$ with identity element $e \in G$, if $x \in G$ then the inverse element for x is the element $x^{-1} \in G$ such that

$$x \cdot x^{-1} = x^{-1} \cdot x = e.$$

In a loop $(G, \cdot)$ with identity element e, the left inverse element of $x \in G$ is the element $x^\lambda \in G$ such that

$$x^\lambda \cdot x = e$$

while the right inverse element of $x \in G$ is the element $x^\rho \in G$ such that

$$x \cdot x^\rho = e$$

In case $(G, \cdot)$ is a quasigroup, then $(G, \cdot)$ is called a left inverse property quasigroup(LIPQ) if it has the left inverse property(LIP) i.e if there exists a bijection

$$J_\lambda \ : \ x \mapsto x^\lambda \text{ on } G \text{ such that } x^\lambda \cdot xy = y.$$

Similarly, $(G, \cdot)$ is called a right inverse property quasigroup(RIPQ) if it has the right inverse property(RIP) i.e if there exists a bijection

$$J_\rho \ : \ x \mapsto x^\rho \text{ on } G \text{ such that } yx \cdot x^\rho = y.$$

A quasigroup that is both a LIPQ and a RIPQ is said to have the inverse property(IP) hence called an inverse property quasigroup(IPQ).

The same definitions hold for a loop and such a loop is called a left inverse property loop(LIPL), right inverse property loop(RIPL) and inverse property loop(IPL) accordingly. J_λ and J_ρ are respectively called the left and right inverse maps. If $J_\lambda = J_\rho$, then we simply write $J = J_\lambda = J_\rho$ and as well we write $x^{-1} = x^\lambda = x^\rho$.

Throughout, we shall employ the use of the bijections;

$$J_\rho \ : \ x \mapsto x^\rho, \ J_\lambda \ : \ x \mapsto x^\lambda, \ L_x \ : \ y \mapsto xy \text{ and } R_x \ : \ y \mapsto yx$$

for a loop and the bijections;

$$J'_\rho \ : \ x \mapsto x^{\rho'}, \ J'_\lambda \ : \ x \mapsto x^{\lambda'}, \ L'_x \ : \ y \mapsto xy \text{ and } R'_x \ : \ y \mapsto yx$$

for its loop isotope. If the identity element of a loop is e then that of the isotope shall be denoted by e'.

There are some classes of loops which do not have the inverse property but have properties which can be considered as variations of the inverse property.

A loop $(G, \cdot)$ is called a weak inverse property loop(WIPL) if and only if it obeys the identity

$$x(yx)^\rho = y^\rho \qquad \text{or} \qquad (xy)^\lambda x = y^\lambda \tag{1.1}$$

for all $x, y \in G$.

A loop $(G, \cdot)$ is called a cross inverse property loop(CIPL) if and only if it obeys the identity

$$xy \cdot x^\rho = y \qquad \text{or} \qquad x \cdot yx^\rho = y \qquad \text{or} \qquad x^\lambda \cdot (yx) = y \qquad \text{or} \qquad x^\lambda y \cdot x = y \tag{1.2}$$

for all $x, y, \in G$.

A loop $(G, \cdot)$ is called an automorphic inverse property loop(AIPL) if and only if it obeys the identity

$$(xy)^{\rho} = x^{\rho}y^{\rho} \qquad \text{or} \qquad (xy)^{\lambda} = x^{\lambda}y^{\lambda} \tag{1.3}$$

for all $x, y, \in G$.

A loop $(G, \cdot)$ is called an anti-automorphic inverse property loop(or AAIPL) if and only if it obeys the identity

$$(xy)^{\rho} = y^{\rho}x^{\rho} \qquad \text{or} \qquad (xy)^{\lambda} = y^{\lambda}x^{\lambda} \tag{1.4}$$

for all $x, y, \in G$.

A loop $(G, \cdot)$ is called a semi-automorphic inverse property loop(SAIPL) if and only if it obeys the identity

$$(xy \cdot x)^{\rho} = x^{\rho}y^{\rho} \cdot x^{\rho} \qquad \text{or} \qquad (xy \cdot x)^{\lambda} = x^{\lambda}y^{\lambda} \cdot x^{\lambda} \tag{1.5}$$

for all $x, y, \in G$.

In the quest for the application of CIPQs with long inverse cycles to cryptography, Keedwell [51] constructed the following CIPQ which we shall specifically call Keedwell CIPQ.

Theorem 1.1.1 *(Keedwell CIPQ)*

Let $(G, \cdot)$ be an abelian group of order n such that $n+1$ is composite. Define a binary operation '$\circ$' on the elements of G by the relation $a \circ b = a^r b^s$, where $rs = n+1$. Then $(G, \circ)$ is a CIPQ and the right crossed inverse of the element a is a^u, where $u = (-r)^3$.

The author also gave examples and detailed explanation and procedures of the use of this CIPQ for cryptography. We shall later introduce the Smarandache Keedwell CIPQ and demonstrate its application to double cryptography.

Quasigroup and loops are known to lack associativity. But some quasigroups and loops obey identities that are called weak-associative laws. Among such identities is the inverse property. Other weak-associative laws shall be introduced under quasigroup and loop varieties and identities.

A quasigroup $(G, \cdot)$ is called a left alternative property quasigroup(LAPQ) if the left alternative property(LAP),

$$xx \cdot y = x \cdot xy$$

holds for all $x, y \in G$.

A quasigroup $(G, \cdot)$ is called a right alternative property quasigroup(RAPQ) if the right alternative property(RAP),

$$y \cdot xx = yx \cdot x$$

holds for all $x, y \in G$.

A quasigroup $(G, \cdot)$ is called an alternative property quasigroup(APQ) if the alternative property(AP) i.e both the LAP and RAP hold.

The same definitions hold for a loop and such a loop is called a left alternative property

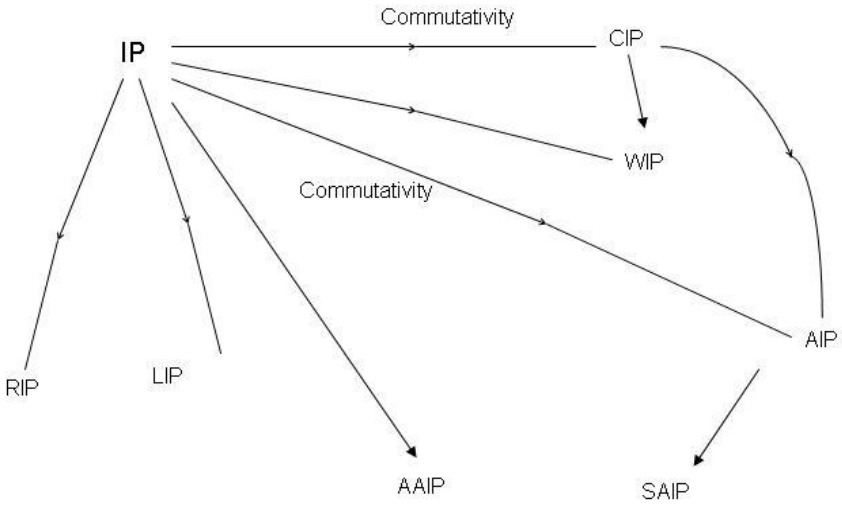


Figure 1.1: Varieties of Inverse Property Loops

loop(LAPL), right alternative property loop(RAPL) and alternative property loop(APL) accordingly.

A loop $(G, \cdot)$ is called a flexible or elastic loop if the flexiblity or elasticity property

$$xy \cdot x = x \cdot yx$$

holds for all $x, y \in G$. $(G, \cdot)$ is said to be a power associative loop if $< x >$ is a subgroup for all $x \in G$ and a diassociative loop if $< x, y >$ is a subgroup for all $x, y \in G$.

1.2 Varieties of Quasigroups And Loops

The varieties of quasigroups and loops can be categorized mainly into two.

(a) Quasigroups and loops of Bol-Moufang type; these are variety of quasigroups and loops defined by a single identity that

- **(i)** involves three distinct variables on both sides,
- **(ii)** contains variables in the same order on both sides
- **(iii)** exactly one of the variables appears twice on both sides.

(b) Quasigroups and loops that are not of Bol-Moufang type; these are variety of quasigroups and loops defined by a single identity or more but which are not of Bol-Moufang type.

1.2.1 Quasigroups And Loops Of Bol-Moufang Types

Fenyves [34] and [33] in the 1960s was the first to classify loops of Bol-Moufang type by showing that there are sixty of them among which thirty are equivalent to the associativity

law. In the beginning of this 21st century, Phillips and Vojtěchovský [73] and [74] generalised and completed the study of Fenyves by showing that there are sixteen varieties of quasigroups and fourteen varieties of loops of Bol-Moufang type. The identities describing the most popular quasigroups and loops of Bol-Moufang are highlighted below. For some particular varieties, the identity or identities named after them are equivalent to each other in quasigroups or in loops or in both quasigroups and loops.

$$(yx \cdot x)z = y(x \cdot xz) \qquad \text{central identity} \tag{1.6}$$

$$(xy \cdot z)x = x(y \cdot zx) \qquad \text{extra identity} \tag{1.7}$$

$$xy \cdot xz = x(yx \cdot z) \qquad \text{extra identity} \tag{1.8}$$

$$yx \cdot zx = (y \cdot xz)x \qquad \text{extra identity} \tag{1.9}$$

$$xx \cdot yz = (x \cdot xy)z \qquad \text{left central identity} \tag{1.10}$$

$$(x \cdot xy)z = x(x \cdot yz) \qquad \text{left central identity} \tag{1.11}$$

$$(xx \cdot y)z = x(x \cdot yz) \qquad \text{left central identity} \tag{1.12}$$

$$(y \cdot xx)z = y(x \cdot xz) \qquad \text{left central identity} \tag{1.13}$$

$$yz \cdot xx = y(zx \cdot x) \qquad \text{right central identity} \tag{1.14}$$

$$(yz \cdot x)x = y(zx \cdot x) \qquad \text{right central identity} \tag{1.15}$$

$$(yz \cdot x)x = y(z \cdot xx) \qquad \text{right central identity} \tag{1.16}$$

$$(yx \cdot x)z = y(xx \cdot z) \qquad \text{right central identity} \tag{1.17}$$

$$xy \cdot zx = (x \cdot yz)x \qquad \text{Moufang identity} \tag{1.18}$$

$$xy \cdot zx = x(yz \cdot x) \qquad \text{Moufang identity} \tag{1.19}$$

$$(xy \cdot x)z = x(y \cdot xz) \qquad \text{Moufang identity} \tag{1.20}$$

$$(yx \cdot z)x = y(x \cdot zx) \qquad \text{Moufang identity} \tag{1.21}$$

$$(x \cdot yx)z = x(y \cdot xz) \qquad \text{left Bol identity} \tag{1.22}$$

$$(yx \cdot z)x = y(xz \cdot x) \qquad \text{right Bol identity} \tag{1.23}$$

So if a quasigroup or loop obeys any of the identities above say identity 'I', then it is called a 'I-quasigroup' or 'I-loop' accordingly.

A loop Q is called a conjugacy closed loop (CC-loop) if and only if the identities

$$x \cdot yz = (xy)/x \cdot xz \text{ and } zy \cdot x = zx \cdot x\backslash(yx)$$

hold in Q.

1.3 Universality Of Loops

Consider $(G, \cdot)$ and $(H, \circ)$ been two distinct groupoids(quasigroups, loops). Let A, B and C be three bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an isotopism of $(G, \cdot)$ onto $(H, \circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

So, $(H, \circ)$ is called a groupoid(quasigroup, loop) isotope of $(G, \cdot)$.

Similarly, the triple

$$\alpha^{-1} = (A, B, C)^{-1} = (A^{-1}, B^{-1}, C^{-1})$$

is an isotopism from $(H, \circ)$ onto $(G, \cdot)$ so that $(G, \cdot)$ is also called a groupoid(quasigroup,loop) isotope of $(H, \circ)$. Hence, both are said to be isotopic to each other.

If one of two isotopic groupoids is a quasigroup, then both are quasigroups, but the same statement is not true if two quasigroups are isotopic and one is a loop. This fact makes it possible and reasonable to study and consider quasigroups as isotopes of groups.

If $C = I$, the identity map on G so that $H = G$, then the triple $\alpha = (A, B, I)$ is called a principal isotopism of $(G, \cdot)$ onto $(G, \circ)$ and $(G, \circ)$ is called a principal isotope of $(G, \cdot)$.

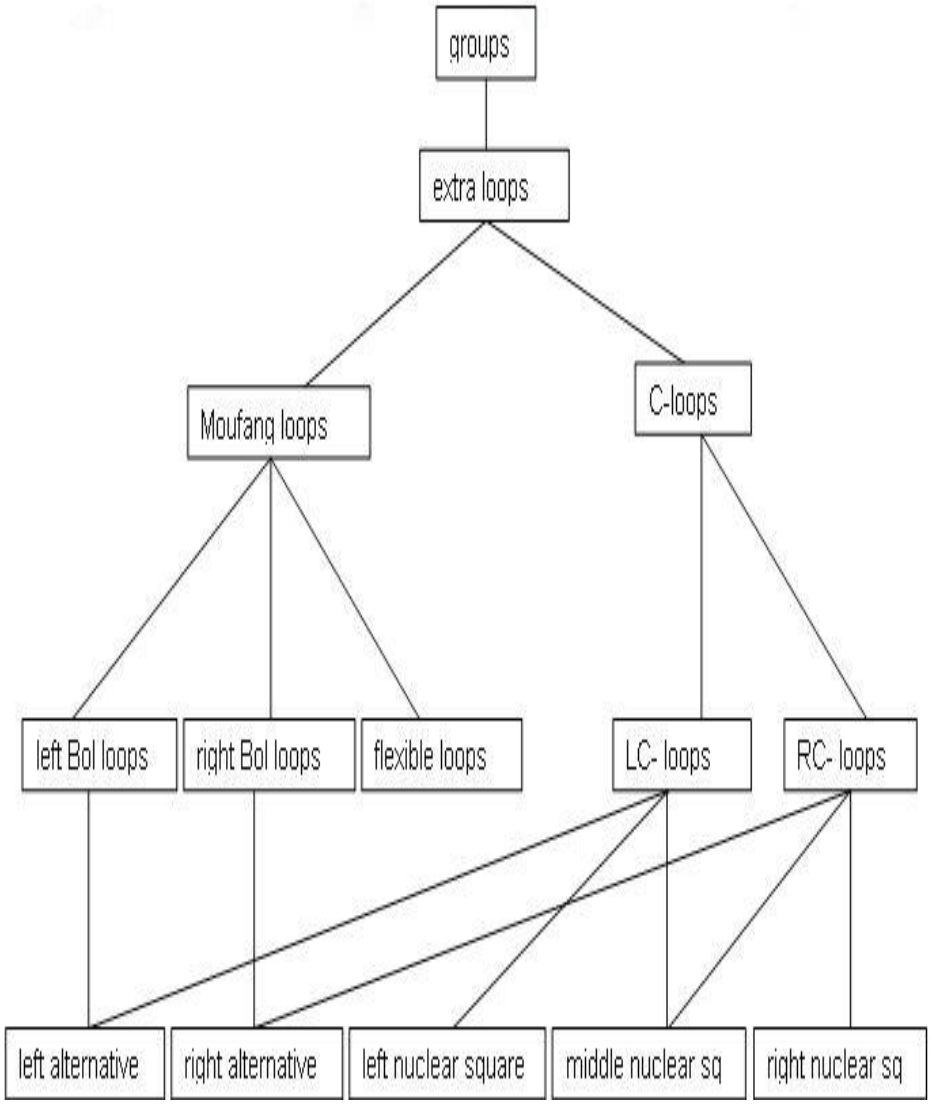


Figure 1.2: Varieties of Loops of Bol-Moufang type

Eventually, the equation of relationship now becomes

$$x \cdot y = xA \circ yB \ \forall \ x, y \in G$$

which is easier to work with. But taken $A = R_g$ and $B = L_f$ for some $f, g \in G$, the relationship now becomes

$$x \cdot y = xR_g \circ yL_f \ \forall \ x, y \in G$$

or

$$x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x, y \in G.$$

With this new form, the triple $\alpha = (R_g, L_f, I)$ is called an f, g-principal isotopism of $(G, \cdot)$ onto $(G, \circ)$, f and g are called translation elements of G or at times written in the pair form (g, f), while $(G, \circ)$ is called an f, g-principal isotope of $(G, \cdot)$.

The last form of α above gave rise to an important result in the study of loop isotopes of loops.

Theorem 1.3.1 *(Bruck [21])*

Let $(G, \cdot)$ and $(H, \circ)$ be two distinct isotopic loops. For some $f, g \in G$, there exists an f, g-principal isotope $(G, *)$ of $(G, \cdot)$ such that $(H, \circ) \cong (G, *)$.

With this result, to investigate the isotopic invariance of an isomorphic invariant property in loops, one simply needs only to check if the property in consideration is true in all f, g-principal isotopes of the loop. A property is isotopic invariant if whenever it holds in the domain loop i.e $(G, \cdot)$ then it must hold in the co-domain loop i.e $(H, \circ)$ which is an isotope of the former. In such a situation, the property in consideration is said to be a universal property hence the loop is called a universal loop relative to the property in consideration as often used by Nagy and Strambach [69] in their algebraic and geometric study of the universality of some types of loops. For instance, if every isotope of "certain" loop is a

"certain" loop, then the formal is called a universal "certain" loop. So, we can now restate Theorem 1.3.1 as :

Theorem 1.3.2 *Let $(G, \cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G, \cdot)$ is a universal "certain" loop if and only if every f, g-principal isotope $(G, *)$ of $(G, \cdot)$ has the "certain" loop property.*

The procedure that has been described so far is called isotopy-isomorphy of loops. It has been used to study the universality of Bol-Moufang type loops such as Bol loops and Moufang loops in [72] which are indeed universal. This approach was difficult for the study of the universality of central loops in Jaíyéọlá [45], so it resulted into the use of left and right translations combined with autotopic characterization and these gave success although not all central loops were found to be universal. We shall later on prove the Smarandache versions of Theorem 1.3.1 and Theorem 1.3.2 for Smarandache loops.

1.3.1 Universality Of Bol-Moufang Type Loops

The isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [34] and [33] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [73], [74] and [64] have been of interest to researchers in loop theory in the recent past. Among such is Etta Falconer's Ph.D [31] and her paper [32] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered.

1.4 Autotopisms Of A Loop

From the earlier discussions, if $(H, \circ) = (G, \cdot)$ then the triple $\alpha = (A, B, C)$ is called an autotopism where $A, B, C \in SYM(G, \cdot)$. Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. If $A = B = C$ then A is an automorphism and the group such bijections form is called the automorphism group $AUM(G, \cdot)$ of $(G, \cdot)$. Chiboka and Solarin [27] solved an open problem by characterizing G-loops by autotopisms. If

$$(UL_c, U, UL_c) \in AUT(G, \cdot) \text{ for some } c \in G,$$

then U is called a left pseudo-automorphism of G with companion c. However, if

$$(V, VR_c, VR_c) \in AUT(G, \cdot) \text{ for some } c \in G,$$

then V is called a right pseudo-automorphism of G with companion c. Such Us and Vs form groups called the left pseudo-automorphism group $PS_\lambda(G, \cdot)$ and right pseudo-automorphism group $PS_\rho(G, \cdot)$ respectively.

1.5 G-Loops

G-loops are loops that are isomorphic to all their loop isotopes, hence if they have a "certain" isomorphic invariant property, they will be universal relative to that property. This is the case of extra loops, they are Moufang loops and conjugacy closed loops(CC-loops introduced by Goodaire and Robinson [38] and [39]), the reverse is also true as well, hence since CC-loops are G-loops, extra loops are G-loops so they are universal Moufang loops since they are Moufang loops and Moufang loops are universal.

Theorem 1.5.1 *(Chiboka and Solarin [27], Kunen [61])*

Let $(G, \cdot)$ be a loop. G is a G-loop if and only if there exists $\theta \in SYM(G, \cdot)$ such that

$$(\theta R_x^{-1}, \theta L_y^{-1}, \theta) \in AUT(G, \cdot) \ \forall \ x, y \in G.$$

1.5.1 Special Mappings Of A Loop

Definition 1.5.1 *(Robinson [77])*

Let $(G, \cdot)$ be a loop. A mapping $\theta \in SYM(G, \cdot)$ is a special map for G means that there exist $f, g \in G$ so that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot).$$

From Definition 1.5.1, it can be observed that θ is a special map for a loop $(G, \cdot)$ if and only if θ is an isomorphism of $(G, \cdot)$ onto some f, g-principal isotope $(G, \circ)$ of $(G, \cdot)$. This is clearly seen because ;

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I) \in AUT(G, \cdot).$$

But since

$$(R_g^{-1}, L_f^{-1}, I) \ : \ (G, \circ) \to (G, \cdot),$$

then for

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot),$$

we must have

$$(\theta, \theta, \theta) \ : \ (G, \cdot) \to (G, \circ)$$

which means $(G, \cdot) \overset{\theta}{\cong} (G, \circ)$.

The importance of special maps can clearly be seen, for their existence is another form of isotopy-isomorphy property for loops. The autotopism in Definition 1.5.1 looks exactly like that in Theorem 1.5.1. We shall be discussing about Smarandache special maps later.

1.6 Bryant-Schneider Group Of A Loop

Robinson [77] went further to show that if

$$BS(G,\cdot) = \{\theta \in SYM(G) \ : \ \exists \ f,g \in G \ \ni \ (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G,\cdot) \leq SYM(G)$ called the Bryant-Schneider group of the loop $(G,\cdot)$ because its importance and motivation stem from the work of Bryant and Schneider [23]. In fact, he established that if $(G,\circ)$ is an f,g-principal isotope of $(G,\cdot)$, then $BS(G,\cdot) = BS(G,\circ)$. Actually, as shown in Bruck [72], for a loop $(G,\cdot)$ with left multiplication group $\mathcal{M}\text{ult}_\lambda(G,\cdot)$, right multiplication group $\mathcal{M}\text{ult}_\rho(G,\cdot)$ and multiplication group $\mathcal{M}\text{ult}(G,\cdot)$, if a loop $(H,\circ)$ is an isotope of $(G,\cdot)$, then

$$\mathcal{M}\text{ult}_\lambda(G,\cdot) \cong \mathcal{M}\text{ult}_\lambda(H,\circ), \ \mathcal{M}\text{ult}_\rho(G,\cdot) \cong \mathcal{M}\text{ult}_\rho(H,\circ) \text{ and } \mathcal{M}\text{ult}(G,\cdot) \cong \mathcal{M}\text{ult}(H,\circ).$$

But if $H = G$ and the isotopism is an f,g-principal isotopism, then

$$\mathcal{M}\text{ult}_\lambda(G,\cdot) = \mathcal{M}\text{ult}_\lambda(H,\circ), \ \mathcal{M}\text{ult}_\rho(G,\cdot) = \mathcal{M}\text{ult}_\rho(H,\circ) \text{ and } \mathcal{M}\text{ult}(G,\cdot) = \mathcal{M}\text{ult}(H,\circ).$$

We call

$$\mathcal{M}\text{ult}_\mu(G,\cdot) = \Big\langle \{R_x, L_x \ : \ x \in G\} \Big\rangle$$

the middle multiplication group. Since the advent of the Bryant-Schneider group, some studies by Adeniran [2, 6, 3, 4, 5] and Chiboka [28] have been done on it relative to CC-loops, Bol loops and extra loops. We shall later on introduce Smarandache Bryant-Schneider group and carry out a study on it for Smarandache loops.

1.7 Smarandache Quasigroups And Loops

The study of Smarandache loops was initiated by W.B. Vasantha Kandasamy in 2002. In her book [86], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [87], she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops and Bruck loops. But in our study here, Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops will be introduced and studied relative to the holomorphs of loops.

We shall also the study new concepts in Smarandache quasigroups after their introduction in Muktibodh [67] and [68].

For this study: a K-loop is an A-loop with the AIP, a Bruck loop is a Bol loop with the AIP, a Burn loop is Bol loop with the conjugacy closed property, an homogeneous loop is an A-loop with the IP and a Kikkawa loop is an A-loop with the IP and AIP.

If there exists at least a non-empty and non-trivial subset M of a groupoid(quasigroup or semigroup or loop) L such that $(M, \cdot)$ is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of $(L, \cdot)$, then L is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ M.

A quasigroup(loop) is called a Smarandache "certain" quasigroup(loop) if it has at least a non-trivial subquasigroup(subloop) with the "certain" property and the latter is referred to as the Smarandache "certain" subquasigroup(subloop). For example, a loop is called a Smarandache Bol-loop if it has at least a non-trivial subloop that is a Bol-loop and the latter is referred to as the Smarandache Bol-subloop. By an "initial S-quasigroup" L with

an "initial S-subquasigroup" L', we mean L and L' are pure quasigroups, i.e. they do not obey a "certain" property(not of any variety).

Definition 1.7.1 *A loop is called a Smarandache left inverse property loop (SLIPL) if it has at least a non-trivial subloop with the LIP.*

A loop is called a Smarandache right inverse property loop (SRIPL) if it has at least a non-trivial subloop with the RIP.

A loop is called a Smarandache inverse property loop (SIPL) if it has at least a non-trivial subloop with the IP.

A loop is called a Smarandache right Bol-loop (SRBL) if it has at least a non-trivial subloop that is a right Bol(RB)-loop.

A loop is called a Smarandache left Bol-loop (SLBL) if it has at least a non-trivial subloop that is a left Bol(LB)-loop.

A loop is called a Smarandache weak inverse property loop (SWIPL) if it has at least a non-trivial subloop with the WIP.

A loop is called a Smarandache G-loop (SG-loop) if it has at least a non-trivial subloop that is a G-loop.

A loop is called a Smarandache CC-loop (SCCL) if it has at least a non-trivial subloop that is a CC-loop.

A loop is called a Smarandache Bol-loop (SBL) if it has at least a non-trivial subloop that is a Bol-loop.

A loop is called a Smarandache central-loop (SCL) if it has at least a non-trivial subloop that is a central-loop.

A loop is called a Smarandache extra-loop (SEL) if it has at least a non-trivial subloop that is a extra-loop.

A loop is called a Smarandache A-loop (SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache K-loop (SKL) if it has at least a non-trivial subloop that is a K-loop.

A loop is called a Smarandache Moufang-loop (SML) if it has at least a non-trivial subloop that is a Moufang-loop.

A loop is called a Smarandache Bruck-loop (SBRL) if it has at least a non-trivial subloop that is a Bruck-loop.

A loop is called a Smarandache Kikkawa-loop (SKWL) if it has at least a non-trivial subloop that is a Kikkawa-loop.

A loop is called a Smarandache Burn-loop (SBNL) if it has at least a non-trivial subloop that is a Burn-loop.

A loop is called a Smarandache homogeneous-loop (SHL) if it has at least a non-trivial subloop that is a homogeneous-loop.

Chapter 2

HOLOMORPHY OF SMARANDACHE QUASIGROUPS AND LOOPS

2.1 Holomorph And Smarandache Holomorph

To every loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define '$\circ$' on H such that

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H,$$

then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [19] and is called the Holomorph of $(L, \cdot)$.

Let $(L, \cdot)$ be an S-quasigroup(S-loop) with S-subgroup $(L', \cdot)$. Define a Smarandache automorphism of L to be the elements of the set

$$SAUM(L) = SAUM(L, \cdot) = \{\alpha \in AUM(L) \ : \ \alpha \ : \ L' \to L'\}.$$

It is easy to see that $SAUM(L) \leq AUM(L)$. So, $SAUM(L)$ will be called the Smarandache automorphism group(SAG) of L. $SAUM(L)$ was called the group of Smarandache loop automorphisms on L' in [86]. Now, set $H_S = (L, \cdot) \times SAUM(L, \cdot)$. If we define '$\circ$' on H_S such that

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H_S,$$

then $H_S(L, \cdot) = (H_S, \circ)$ is a S-quasigroup(S-loop) with S-subgroup $(H', \circ)$ where $H' = L' \times SAUM(L)$ and thus will be called the Smarandache Holomorph(SH or S-holomorph) of $(L, \cdot)$. If L is a S-groupoid with a S-subsemigroup H, then the set $SSYM(L, \cdot) = SSYM(L)$ of all bijections A in L such that $A : H \to H$ forms a group called the Smarandache permutation(symmetric) group of the S-groupoid. In fact, $SSYM(L) \leq SYM(L)$.

If in L,

$$x^{-1} \cdot x\alpha \in N(L) \text{ or } x\alpha \cdot x^{-1} \in N(L) \text{ for all } x \in L \text{ and } \alpha \in AUM(L, \cdot),$$

$(H, \circ)$ is called a Nuclear-holomorph of L, if

$$x^{-1} \cdot x\alpha \in C(L) \text{ or } x\alpha \cdot x^{-1} \in C(L) \text{ for all } x \in L \text{ and } \alpha \in AUM(L, \cdot),$$

$(H, \circ)$ is called a Centrum-holomorph of L hence a Central-holomorph if

$$x^{-1} \cdot x\alpha \in Z(L) \text{ or } x\alpha \cdot x^{-1} \in Z(L) \text{ for all } x \in L \text{ and } \alpha \in AUM(L, \cdot).$$

The left Smarandache nucleus of L denoted by $SN_\lambda(L, \cdot) = N_\lambda(L, \cdot) \cap H$. The right Smarandache nucleus of L denoted by $SN_\rho(L, \cdot) = N_\rho(L, \cdot) \cap H$. The middle Smarandache nucleus of L denoted by $SN_\mu(L, \cdot) = N_\mu(L, \cdot) \cap H$. The Smarandache nucleus of L denoted by $SN(L, \cdot) = N(L, \cdot) \cap H$. The Smarandache centrum of L denoted by $SC(L, \cdot) = C(L, \cdot) \cap H$.

The Smarandache center of L denoted by $SZ(L,\cdot) = Z(L,\cdot) \cap H$.

If in L,

$$s^{\lambda} \cdot s\alpha \in SN(L) \text{ or } s\alpha \cdot s^{\rho} \in SN(L) \text{ for all } s \in H \text{ and } \alpha \in SAUM(L,\cdot),$$

$(H_S, \circ)$ is called a Smarandache Nuclear-holomorph of L, if

$$s^{\lambda} \cdot s\alpha \in SC(L) \text{ or } s\alpha \cdot s^{\rho} \in SC(L) \text{ for all } s \in H \text{ and } \alpha \in SAUM(L,\cdot),$$

$(H_S, \circ)$ is called a Smarandache Centrum-holomorph of L hence a Smarandache Central-holomorph if

$$s^{\lambda} \cdot s\alpha \in SZ(L) \text{ or } s\alpha \cdot s^{\rho} \in SZ(L) \text{ for all } s \in H \text{ and } \alpha \in SAUM(L,\cdot).$$

Interestingly, Adeniran [1] and Robinson [75], Oyebo and Adeniran [71], Chiboka and Solarin [26], Bruck [19], Bruck and Paige [22], Robinson [76], Huthnance [40] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [40] showed that if $(L,\cdot)$ is a loop with holomorph $(H,\circ)$, $(L,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

Theorem 2.1.1 *(Adeniran [1] and Robinson [75])*

Let $(L,\cdot)$ be a loop with nuclear holomorph $H(L)$. L is a Bol loop if and only if $H(L)$ is a Bol loop.

Theorem 2.1.2 *(Oyebo and Adeniran [71])*

Let $(L,\cdot)$ be a loop with nuclear holomorph $H(L)$. L is a central loop if and only if $H(L)$ is a central loop.

Theorem 2.1.3 *(Robinson [76])*

Let $(L,\cdot)$ be a loop with nuclear holomorph $H(L)$. L is an extra loop if and only if $H(L)$ is an extra loop.

Theorem 2.1.4 *(Chiboka and Solarin [26])*

Let $(L,\cdot)$ be a loop with nuclear holomorph $H(L)$. L is a CC-loop if and only if $H(L)$ is a CC-loop.

Theorem 2.1.5 *(Bruck [19])*

Let $(L,\cdot)$ be a loop with holomorph $H(L)$. L is an IPL if and only if $H(L)$ is an IPL.

Theorem 2.1.6 *(Bruck and Paige [22])*

Let $(L,\cdot)$ be a loop with central holomorph $H(L)$. L is an A-loop if and only if $H(L)$ is an A-loop.

Theorem 2.1.7 *(Huthnance [40])*

Let $(L,\cdot)$ be a loop with holomorph $H(L)$. L is an WIPL if and only if $H(L)$ is an WIPL.

2.2 Smarandache Isotopism And Autotopism

Definition 2.2.1 *Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $A : L' \to G'$, where $A \in \{U,V,W\}$, then the isotopism $(U,V,W) : (L,\cdot) \to (G,\circ)$ is called a Smarandache isotopism(S-isotopism). Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L,\cdot) \succsim (G,\circ)$.*

Similarly, if $(L,\cdot)$ is an S-groupoid with S-subsemigroup L' such that $A \in \{U,V,W\}$ is a Smarandache permutation, then the autotopism (U,V,W) is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L,\cdot)$ which will be called the Smarandache autotopism group of $(L,\cdot)$. Observe that $SAUT(L,\cdot) \leq AUT(L,\cdot)$.

Discussions To be more precise about the notion of S-isotopism in Definition 2.2.1, the following explanations are given. For a given S-groupoid, the S-subsemigroup is arbitrary. But in the proofs, we shall make use of one arbitrary S-subsemigroup for an S-groupoid at a time for our arguments. Now, if $(L, \cdot)$ and $(G, \circ)$ are S-isotopic S-groupoids with arbitrary S-subsemigroups L' and G' respectively under the triple (U, V, W). In case the S-subsemigroup L' of the S-groupoid L is replaced with another S-groupoid L'' of L(i.e a situation where by L has at least two S-subsemigroups), then under the same S-isotopism (U, V, W), the S-groupoid isotope G has a second S-subsemigroups G''. Hence, when studying the S-isotopism (U, V, W), it will be for the system

$$\{(L, \cdot), (L', \cdot)\} \to \{(G, \circ), (G', \circ)\} \text{ or } \{(L, \cdot), (L'', \cdot)\} \to \{(G, \circ), (G'', \circ)\}$$

and not

$$\{(L, \cdot), (L', \cdot)\} \to \{(G, \circ), (G'', \circ)\} \text{ or } \{(L, \cdot), (L'', \cdot)\} \to \{(G, \circ), (G', \circ)\}.$$

This is because $|L'| = |G'|$ and $|L''| = |G''|$ since $(L')A = G'$ and $(L'')A = G''$ for all $A \in \{U, V, W\}$ while it is not compulsory that $|L'| = |G''|$ and $|L''| = |G'|$. It is very easy to see from the definition that the component transformations U, V, W of isotopy after restricting them to the S-subsemigroup or S-subgroup L' are bijections. Let $x_1, x_2 \in L'$, then $x_1A = x_2A$ implies that $x_1 = x_2$ because $x_1, x_2 \in L'$ implies $x_1, x_2 \in L$, hence $x_1A = x_2A$ in L implies $x_1 = x_2$. The mappings $A : L' \to G'$ and $A : L - L' \to G - G'$ are bijections because $A : L \to G$ is a bijection. Our explanations above are illustrated with the following examples.

Example 2.2.1 *The systems* $(L, \cdot)$ *and* $(L, *)$, $L = \{0, 1, 2, 3, 4\}$ *with the multiplication Table 2.1 and Table 2.2 are S-quasigroups with S-subgroups* $(L', \cdot)$ *and* $(L'', *)$ *respectively,* $L' = \{0, 1\}$ *and* $L'' = \{1, 2\}$. $(L, \cdot)$ *is taken from Example 2.2 of [68]. The triple* (U, V, W)

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

Table 2.1: A Smarandache Quasigroup

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Table 2.2: A Smarandache Quasigroup Isotope

such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L, \cdot)$ onto $(L, *)$. Notice that $A(L') = L''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L''$ are all bijcetions.

Example 2.2.2 *According to Example 4.2.2 of [89], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication Table 2.3. The triple (U, V, W) such that*

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Table 2.3: A Smarandache Semigroup

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

Table 2.4: A Smarandache Semigroup

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in Table 2.4. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijcetions.

From Example 2.2.1 and Example 2.2.2, it is very clear that the study of of S-isotopy of two S-groupoids or S-quasigroups or S-semigroups or S-loops is independent of the S-subsemigroup or S-subgroup that is in consideration. All results in this paper are true for any given S-subsemigroups or S-subgroups of two S-isotopic S-groupoids or S-quasigroups or

S-semigroups or S-loops. More examples of S-isotopic S-groupoids can be constructed using S-groupoids in [88].

Remark 2.2.1 *Taking careful look at Definition 2.2.1 and comparing it with [Definition 4.4.1,[86]], it will be observed that the author did not allow the component bijections U,V and W in (U, V, W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [86] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.*

2.3 Holomorphy Of Some Smarandache Loops

Theorem 2.3.1 *Let $(L, \cdot)$ be a Smarandanche loop with subgroup $(S, \cdot)$. The holomorph $H(S)$ of S is a group.*

Theorem 2.3.2 *A loop is a Smarandache loop if and only if its holomorph is a Smarandache loop.*

Proof

Let L be a Smarandache loop with subgroup S. By Theorem 2.3.1, $(H(S), \circ)$ is a group where $H(S) = AUM(S, \cdot) \times (S, \cdot)$. Clearly, $H(S) \not\subset H(L, \cdot)$. So, let us replace $AUM(S, \cdot)$ in $H(S)$ by $SAUM(L, \cdot)$. $SAUM(L, \cdot) \leq AUM(L, \cdot)$ hence, $H(S) = SAUM(L, \cdot) \times (S, \cdot)$ remains a group. In fact, $(H(S), \circ) \subset (H, \circ)$ and $(H(S), \circ) \leq (H, \circ)$. Thence, the holomorph of a Smarandache loop is a Smarandache loop.

To prove the converse, recall that $H(L) = AUM(L) \times L$. If $H(L)$ is a Smarandache loop then $\exists\ S_H \subset H(L)\ \ni\ S_H \leq H(L)$.

$$S_H \subset H(L) \Rightarrow\ \exists\ BUM(L) \subset AUM(L) \text{ and } B \subset L\ \ni\ S_H = BUM(L) \times B.$$

Let us choose

$$BUM(L) = \{\alpha \in AUM(L)\ :\ b\alpha \in B\ \forall\ b \in B\},$$

this is the group of Smarandache loop automorphisms on B. So, $(S_H, \circ) = (BUM(L) \times B, \circ)$ is expected to be a group. Thus,

$$(\alpha, x) \circ [(\beta, y) \circ (\gamma, z)] = [(\alpha, x) \circ (\beta, y)] \circ (\gamma, z)\ \forall\ x, y, z \in B,\ \alpha, \beta, \gamma \in Bum(L)$$
$$\Leftrightarrow x\beta\gamma \cdot (y\gamma \cdot z) = (x\beta\gamma \cdot y\gamma) \cdot z \Leftrightarrow$$
$$x' \cdot (y' \cdot z) = (x' \cdot y') \cdot z\ \forall\ x', y', z \in B.$$

So, $(B, \cdot)$ must be a group. Hence, L is a Smarandache loop.

Remark 2.3.1 *It must be noted that if $AUM(L, \cdot) = SAUM(L, \cdot)$, then S is a characteristic subloop. A subloop S of a loop L is said to be characteristic if it is its own automorphic image under any automorphism of the loop L.*

Theorem 2.3.3 *Let L and L' be loops. $L \cong L'$ implies $H(L) \cong H(L')$.*

Proof

If $L \cong L'$ then we need to find a bijection ϕ such that

$$AUM(L)) \overset{\phi}{\cong} AUM(L') \Rightarrow H(L) = AUM(L) \times L \cong AUM(L') \times L' = H(L').$$

$$\text{Already, } \alpha\ :\ L \to L'\ \ni\ (\alpha, \alpha, \alpha)\ :\ L \to\ L'$$

is an isotopism. According to [72], if two loops are isotopic, then their groups of autotopism are isomorphic. The automorphism group is one of such since it is a form of autotopism. Thus;

$$AUM(L) \cong AUM(L') \Rightarrow H(L) = AUM(L) \times L \cong AUM(L') \times L' = H(L').$$

Theorem 2.3.4 *Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups such that $SAUM(U)$ and $SAUM(V)$ are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $H_S(U) \gtrsim H_S(V)$ if and only if*

$$x\delta \otimes y\gamma = (x\beta \oplus y)\delta \ \forall \ x, y \in L, \ \beta \in SAUM(U) \text{ and some } \delta, \gamma \in SAUM(V).$$

Hence:

1. *$\gamma \in SAUM(U)$ if and only if $(I, \gamma, \delta) \in SAUT(V)$.*
2. *if U is a initial S-loop, then;*
   (a) *$\mathcal{L}_{e\delta} \in SAUM(V)$.*
   (b) *$\beta \in SAUM(V)$ if and only if $\mathcal{R}_{e\gamma} \in SAUM(V)$.*

   where e is the identity element in U and $\mathcal{L}_x$, $\mathcal{R}_x$ are respectively the left and right translations mappings of $x \in V$.
3. *if $\delta = I$, then $|SAUM(U)| = |SAUM(V)| = 3$ and so $SAUM(U)$ and $SAUM(V)$ are boolean groups.*
4. *if $\gamma = I$, then $|SAUM(U)| = |SAUM(V)| = 1$.*

Proof

Let $H_S(L,\oplus) = (H_S,\circ)$ and $H_S(L,\otimes) = (H_S,\odot)$. $H_S(U) \succsim H_S(V)$ if and only if there exists a bijection $\phi \ : \ H_S(U) \to H_S(V)$ such that

$$[(\alpha,x)\circ(\beta,y)]\phi = (\alpha,x)\phi \odot (\beta,y)\phi \text{ and } (H',\oplus) \overset{\phi}{\cong} (H'',\otimes)$$

where $H' = L' \times SAUM(U)$ and $H'' = L'' \times SAUM(V)$, $(L',\oplus)$ and $(L'',\otimes)$ been the initial S-subquasigroups of U and V. Define

$$(\alpha,x)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \ \forall \ (\alpha,x) \in (H_S,\circ) \text{ where } \psi \in SSYM(L).$$

$$H_S(U) \cong H_S(V) \Leftrightarrow (\alpha\beta, x\beta\oplus y)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi)\odot(\psi^{-1}\beta\psi, y\psi^{-1}\beta\psi)$$
$$\Leftrightarrow (\psi^{-1}\alpha\beta\psi, (x\beta\oplus y)\psi^{-1}\alpha\beta\psi) = (\psi^{-1}\alpha\beta\psi, x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi) \Leftrightarrow$$
$$(x\beta\oplus y)\psi^{-1}\alpha\beta\psi = x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi \Leftrightarrow x\delta \otimes y\gamma = (x\beta\oplus y)\delta$$

$$\text{where } \delta = \psi^{-1}\alpha\beta\psi, \ \gamma = \psi^{-1}\beta\psi.$$

Note that,

$$\gamma\mathcal{L}_{x\delta} = L_{x\beta}\delta \text{ and } \delta\mathcal{R}_{y\gamma} = \beta R_y\delta \ \forall \ x,y \in L.$$

So, when U is an S-loop,

$$\gamma\mathcal{L}_{e\delta} = \delta \text{ and } \delta\mathcal{R}_{e\gamma} = \beta\delta.$$

These can easily be used to prove the remaining part of the theorem.

Corollary 2.3.1 *Let $U = (L,\oplus)$ and $V = (L,\otimes)$ be initial S-quasigroups. $H_S(U) \succsim H_S(V)$ implies U and V are S-isotopic under a triple of the form (δ, I, δ).*

Proof

In Theorem 2.3.4, let $\beta = I$, then $\gamma = I$. The conclusion follows immediately.

Remark 2.3.2 *By Theorem 2.3.3 and Corollary 2.3.1, any two distinct S-isomorphic S-loops with the same underlining set are non-trivially S-isotopic.*

Corollary 2.3.2 *Let L be a Smarandache loop. If L is isomorphic to L', then $\{H(L), H(L')\}$ and $\{L, L'\}$ are both systems of isomorphic Smarandache loops.*

Proof

This follows from Theorem 2.3.2, Theorem 2.3.3 and the obvious fact that the Smarandache loop property in loops is isomorphic invariant.

Remark 2.3.3 *The fact in Corollary 2.3.2 that $H(L)$ and $H(L')$ are isomorphic Smarandache loops could be a clue to solve one of the problems posed in [87]. The problem required us to prove or disprove that every Smarandache loop has a Smarandache loop isomorph.*

2.3.1 Holomorphy Of Smarandache Inverse Property Loops

Theorem 2.3.5 *Let L be a loop with holomorph $H(L)$. L is an IP-SIPL if and only if $H(L)$ is an IP-SIPL.*

Proof

In an IPL, every subloop is an IPL. So if L is an IPL, then it is an IP-SIPL. From Theorem 2.1.5, it was stated that L is an IPL if and only if $H(L)$ is an IPL. Hence, $H(L)$ is an IP-SIPL. Conversely assuming that $H(L)$ is an IP-SIPL and using the same argument, L is an IP-SIPL

Theorem 2.3.6 *Let L be a loop with S-holomorph $H_S(L)$. L is a SIPL if and only if $H_S(L)$ is a SIPL.*

Proof

Assume that L is a SIPL and let L' be the S-IP subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.5 that $(H', \circ)$ is an IPL, hence, a S-IP subloop of H_S. Thus, $H_S(L)$ is a SIPL. Conversely, if $H_S(L)$ is a SIPL, then it has a S-IP subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is an IPL since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SIPL.

Theorem 2.3.7 *Let L be a loop with holomorph $H(L)$. L is an WIP-SWIPL if and only if $H(L)$ is an WIP-SWIPL.*

Proof

In a WIPL, every subloop is a WIPL. So if L is a WIPL, then it is a WIP-SWIPL. From Theorem 2.1.7, it can be stated that L is a WIPL if and only if $H(L)$ is a WIPL. Hence, $H(L)$ is a WIP-SWIPL. Conversely assuming that $H(L)$ is a WIP-SWIPL and using the same argument L is a WIP-SWIPL.

Theorem 2.3.8 *Let L be a loop with S-holomorph $H_S(L)$. L is a SWIPL if and only if $H_S(L)$ is a SWIPL.*

Proof

Assume that L is a SWIPL and let L' be the S-WIP subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.7 that $(H', \circ)$ is an WIPL, hence, a S-WIP subloop of H_S. Thus, $H_S(L)$ is a SWIPL. Conversely, if $H_S(L)$ is a SWIPL, then it has a S-WIP subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a WIPL since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SWIPL.

Theorem 2.3.9 *Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is an AIPQ(AIPL) if and only if*

1. *$AUM(L)$ is an abelian group,*

2. *$(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and*

3. *L is a AIPQ(AIPL).*

Proof

A quasigroup(loop) is an automorphic inverse property loop(AIPL) if and only if it obeys the AIP identity. Using either of the definitions of an AIPQ(AIPL), it can be shown that $H(L)$ is a AIPQ(AIPL) if and only if $AUM(L)$ is an abelian group and

$$(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L).$$

L is isomorphic to a subquasigroup(subloop) of $H(L)$, so L is a AIPQ(AIPL) which implies $(J_\rho, J_\rho, J_\rho) \in AUT(L)$. So,

$$(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L).$$

Corollary 2.3.3 *Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if*

1. *$AUM(L)$ is an abelian group,*

2. *$(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and*

3. *L is a CIPQ(CIPL).*

Proof

A quasigroup(loop) is a CIPQ(CIPL) if and only if it is a WIPQ(WIPL) and an AIPQ(AIPL). L is a WIPQ(WIPL) if and only if $H(L)$ is a WIPQ(WIPL).

If $H(L)$ is a CIPQ(CIPL), then $H(L)$ is both a WIPQ(WIPL) and a AIPQ(AIPL) which implies 1., 2., and 3. of Theorem 2.3.9. Hence, L is a CIPQ(CIPL). The converse follows by just doing the reverse.

Corollary 2.3.4 *Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is an AIPQ(AIPL) or CIPQ(CIPL), then $H(L) \cong L$.*

Proof

By 2. of Theorem 2.3.9,

$$(\beta^{-1}, \alpha, I) \in AUT(L) \; \forall \; \alpha, \beta \in AUM(L)$$

implies $x\beta^{-1} \cdot y\alpha = x \cdot y$ which means $\alpha = \beta = I$ by substituting $x = e$ and $y = e$. Thus, $AUM(L) = \{I\}$ and so $H(L) \cong L$.

Theorem 2.3.10 *The holomorph of a quasigroup(loop) L is a AIPQ(AIPL) or CIPQ(CIPL) if and only if $AUM(L) = \{I\}$ and L is a AIPQ(AIPL) or CIPQ(CIPL).*

Proof

This is established using Theorem 2.3.9, Corollary 2.3.3 and Corollary 2.3.4.

Theorem 2.3.11 *Let $(L, \cdot)$ be a quasigroups(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if $AUM(L)$ is an abelian group and any of the following is true for all $x, y \in L$ and $\alpha, \beta \in AUM(L)$:*

1. $(x\beta \cdot y)x^{\rho} = y\alpha$.
2. $x\beta \cdot yx^{\rho} = y\alpha$.
3. $(x^{\lambda}\alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.
4. $x^{\lambda}\alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Proof

This is achieved by simply using the four equivalent identities that define a CIPQ(CIPL):

Corollary 2.3.5 *Let $(L, \cdot)$ be a quasigroups(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, the following are equivalent to each other*

1. $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.
2. $(\beta^{-1}J_\lambda, \alpha J_\lambda, J_\lambda) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.
3. $(x\beta \cdot y)x^\rho = y\alpha$.
4. $x\beta \cdot yx^\rho = y\alpha$.
5. $(x^\lambda \alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.
6. $x^\lambda \alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Hence,

$$(\beta, \alpha, I), (\alpha, \beta, I), (\beta, I, \alpha), (I, \alpha, \beta) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L).$$

Proof

The equivalence of the six conditions follows from Theorem 2.3.11 and the proof of Theorem 2.3.9. The last part is simple.

Corollary 2.3.6 *Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, L is a flexible unipotent CIPQ(flexible CIPL of exponent 2).*

Proof

It is observed that $J_\rho = J_\lambda = I$. Hence, the conclusion follows.

Remark 2.3.4 *The holomorphic structure of loops such as extra loop, Bol-loop, C-loop, CC-loop and A-loop have been found to be characterized by some special types of automorphisms such as*

1. *Nuclear automorphism(in the case of Bol-,CC- and extra loops),*

2. *central automorphism(in the case of central and A-loops).*

By Theorem 2.3.9 and Corollary 2.3.3, the holomorphic structure of AIPLs and CIPLs is characterized by commutative automorphisms.

Theorem 2.3.12 *The holomorph $H(L)$ of a quasigroup(loop) L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL) if and only if $SAUM(L) = \{I\}$ and L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL).*

Proof

Let L be a quasigroup with holomorph $H(L)$. If $H(L)$ is a SAIPQ(SCIPQ), then there exists a S-subquasigroup $H'(L) \subset H(L)$ such that $H'(L)$ is a AIPQ(CIPQ). Let $H'(L) = G \times SAUM(L)$ where G is the S-subquasigroup of L. From Theorem 2.3.10, it can be seen that $H'(L)$ is a AIPQ(CIPQ) if and only if $SAUM(L) = \{I\}$ and G is a AIPQ(CIPQ). So the conclusion follows.

2.3.2 Holomorphy Of Smarandache Conjugacy Closed Loops

Theorem 2.3.13 *Every G-loop is a SG-loop.*

Proof

As shown in [Lemma 2.2, [27]], every subloop in a G-loop is a G-loop. Hence, the claim follows.

Corollary 2.3.7 *CC-loops are SG-loops.*

Proof

In [38], CC-loops were shown to be G-loops. Hence, the result follows by Theorem 2.3.13.

Theorem 2.3.14 *Let G be a CC-loop with normal subloop H. G/H is a SG-loop.*

Proof

According to [Theorem 2.1,[27]], G/H is a G-loop. Hence, by Theorem 2.3.13, the result follows.

Theorem 2.3.15 *Every SCCL is a SG-loop.*

Proof

If a loop L is a SCCL, then there exist a subloop H of L that is a CC-loop. CC-loops are G-loops, hence, H is a G-loop which implies L is a SG-loop.

Theorem 2.3.16 *Every CC-loop is a SCCL.*

Proof

By the definition of CC-loop in [59], [58] and [61], every subloop of a CC-loop is a CC-loop. Hence, the conclusion follows.

Remark 2.3.5 *The fact in Corollary 2.3.7 that CC-loops are SG-loops can be seen from Theorem 2.3.15 and Theorem 2.3.16.*

Theorem 2.3.17 *Let L be a loop with Nuclear-holomorph $H(L)$. L is an IP-CC-SIP-SCCL if and only if $H(L)$ is an IP-CC-SIP-SCCL.*

Proof

If L is an IP-CCL, then by Theorem 2.3.5, $H(L)$ is an IP-SIPL and hence by Theorem 2.1.4 and Theorem 2.3.16, $H(L)$ is an IP-CC-SIP-SCCL. The converse is true by assuming that $H(L)$ is an IP-CC-SIP-SCCL and using the same reasoning.

Theorem 2.3.18 *Let L be a loop with S-nuclear holomorph $H_S(L)$. L is a SCCL if and only if $H_S(L)$ is a SCCL.*

Proof

Assume that L is a SCCL and let L' be the S-CC subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.4 that $(H', \circ)$ is a CCL, hence, a S-CC subloop of H_S. Thus, $H_S(L)$ is a SCCL. Conversely, if $H_S(L)$ is a SCCL, then it has a S-CC subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a CCL since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SCCL.

2.3.3 Holomorphy Of Smarandache: Bol Loops, Central Loops, Extra Loops And Burn Loops

Theorem 2.3.19 *Let L be a loop with Nuclear-holomorph $H(L)$. L is a Bol-SBL if and only if $H(L)$ is a Bol-SBL.*

Proof

If L is a Bol-loop, then by Theorem 2.1.1, $H(L)$ is a Bol-loop. According to [Theorem 6, [87]], every Bol-loop is a SBL. Hence, $H(L)$ is a Bol-SBL. The Converse is true by using the same argument.

Theorem 2.3.20 *Let L be a loop with S-nuclear holomorph $H_S(L)$. L is a SBL if and only if $H_S(L)$ is a SBL.*

Proof

Assume that L is a SBL and let L' be the S-Bol subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.1 that $(H', \circ)$ is a Bol loop, hence, a S-Bol subloop of H_S. Thus, $H_S(L)$ is a SBL. Conversely, if $H_S(L)$ is a SBL, then it has a S-Bol subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a Bol loop since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SBL.

Theorem 2.3.21 *Let L be a loop with Nuclear-holomorph $H(L)$. L is a central-SCL if and only if $H(L)$ is a central-SCL.*

Proof

If L is a central-loop, then by Theorem 2.1.2, $H(L)$ is a central-loop. Every central-loop is a SCL. Hence, $H(L)$ is a central-SCL. The Converse is true by using the same argument.

Theorem 2.3.22 *Let L be a loop with S-nuclear holomorph $H_S(L)$. L is a SCL if and only if $H_S(L)$ is a SCL.*

Proof

Assume that L is a SCL and let L' be the S-central subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.2 that $(H', \circ)$ is a central loop, hence, a S-central subloop of H_S. Thus, $H_S(L)$ is a SCL. Conversely, if $H_S(L)$ is a SCL, then it has a S-central subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a central loop since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SCL.

Theorem 2.3.23 *Let L be a loop with Nuclear-holomorph $H(L)$. L is a extra-SEL if and only if $H(L)$ is an extra-SEL.*

Proof

If L is a extra-loop, then by Theorem 2.1.3, $H(L)$ is a extra-loop. Every extra-loop is a SEL. Hence, $H(L)$ is a extra-SEL. The Converse is true by using the same argument.

Theorem 2.3.24 *Let L be a loop with S-nuclear holomorph $H_S(L)$. L is a SEL if and only if $H_S(L)$ is a SEL.*

Proof

Assume that L is a SEL and let L' be the S-extra subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn

from Theorem 2.1.3 that $(H', \circ)$ is a extra loop, hence, a S-extra subloop of H_S. Thus, $H_S(L)$ is a SEL. Conversely, if $H_S(L)$ is a SEL, then it has a S-extra subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a extra loop since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SEL.

Corollary 2.3.8 *Let L be a loop with Nuclear-holomorph $H(L)$. L is a IP-Burn-SIP-SBNL if and only if $H(L)$ is an IP-Burn-SIP-SBNL.*

Proof

This follows by combining Theorem 2.3.17 and Theorem 2.3.19.

Corollary 2.3.9 *Let L be a loop with S-nuclear holomorph $H_S(L)$. L is a SBNL if and only if $H_S(L)$ is a SBNL.*

Proof

This follows by combining Theorem 2.3.18 and Theorem 2.3.20.

2.3.4 Holomorphy Of Smarandache: A-Loops, Homogeneous Loops

Theorem 2.3.25 *Every A-loop is a SAL.*

Proof

According to [Theorem 2.2, [22]], every subloop of an A-loop is an A-loop. Hence, the conclusion follows.

Theorem 2.3.26 *Let L be a loop with Central-holomorph $H(L)$. L is an A-SAL if and only if $H(L)$ is an A-SAL.*

Proof

If L is an A-loop, then by Theorem 2.1.6, $H(L)$ is a A-loop. By Theorem 2.3.25, every A-loop is a SAL. Hence, $H(L)$ is an A-SAL. The Converse is true by using the same argument.

Theorem 2.3.27 *Let L be a loop with S-central holomorph $H_S(L)$. L is a SAL if and only if $H_S(L)$ is a SAL.*

Proof

Assume that L is a SAL and let L' be the S-extra subloop in L. Then, consider $H' = L' \times SAUM(L) \subset H_S = L \times SAUM(L)$. Recall that $(H_s, \circ)$ is a loop. It can be drawn from Theorem 2.1.6 that $(H', \circ)$ is an A-loop, hence, a S-A-subloop of H_S. Thus, $H_S(L)$ is a SAL. Conversely, if $H_S(L)$ is a SAL, then it has a S-A subloop H'. Say, $H' = L' \times SAUM(L)$ which means L' is a A-loop since $H'' = L' \times \{I\}$ is a subloop of H' and $H'' \cong L'$. Therefore, L is a SAL.

Corollary 2.3.10 *Let L be a loop with Central-holomorph $H(L)$. L is an homogeneous-SHL if and only if $H(L)$ is an homogeneous-SHL.*

Proof

This can be seen by combining Theorem 2.3.5 and Theorem 2.3.26.

Corollary 2.3.11 *Let L be a loop with S-Central-holomorph $H_S(L)$. L is an homogeneous-SHL if and only if $H_S(L)$ is an homogeneous-SHL.*

Proof

This can be seen by combining Theorem 2.3.6 and Theorem 2.3.27.

2.3.5 Holomorphy Of Smarandache: K-Loops, Bruck-Loops and Kikkawa-Loops

Corollary 2.3.12 *The holomorph $H(L)$ of a loop L is a SKL or SBRL or SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKL or SBRL or SKWL.*

Proof

Let L be a loop with holomorph $H(L)$. Consider a subloop $H'(L)$ of $H(L)$ such that $H'(L) = G \times SAUM(L)$ where G is a subloop of L.

1. Recall that by Theorem 2.1.6, $H'(L)$ is an A-loop if and only if it is a Smarandache Central-holomorph of L and G is an A-loop. Combing this fact with Theorem 2.3.12, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKL if and only if $SAUM(L) = \{I\}$ and L is a SKL.

2. Recall that by Theorem 2.1.1, $H'(L)$ is a Bol loop if and only if it is a Smarandache Nuclear-holomorph of L and G is a Bol-loop. Combing this fact with Theorem 2.3.12, it can be concluded that: the holomorph $H(L)$ of a loop L is a SBRL if and only if $SAUM(L) = \{I\}$ and L is a SBRL.

3. Following the first reason in 1., and using Theorem 2.3.12, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKWL.

2.4 Double Cryptography Using The Smarandache Keedwell CIQ

In the quest for the application of CIPQs with long inverse cycles to cryptography, Keedwell [51] constructed the following CIPQ which we shall specifically call Keedwell CIPQ.

Theorem 2.4.1 *(Keedwell CIPQ)*

Let $(G, \cdot)$ be an abelian group of order n such that $n+1$ is composite. Define a binary operation '$\circ$' on the elements of G by the relation $a \circ b = a^r b^s$, where $rs = n+1$. Then $(G, \circ)$ is a CIPQ and the right crossed inverse of the element a is a^u, where $u = (-r)^3$

The author also gave examples and detailed explanation and procedures of the use of this CIPQ for cryptography. Cross inverse property quasigroups have been found appropriate for cryptography because of the fact that the left and right inverses x^λ and x^ρ of an element x

do not coincide unlike in left and right inverse property loops, hence this gave rise to what is called 'cycle of inverses' or 'inverse cycles' or simply 'cycles' i.e finite sequence of elements

$$x_1, x_2, \cdots, x_n \text{ such that } x_k^{\rho} = x_{k+1} \mod n.$$

The number n is called the length of the cycle. The origin of the idea of cycles can be traced back to Artzy [7, 9] where he also found there existence in WIPLs apart form CIPLs. In his two papers, he proved some results on possibilities for the values of n and for the number m of cycles of length n for WIPLs and especially CIPLs. We call these "Cycle Theorems" for now.

In application, it is assumed that the message to be transmitted can be represented as single element x of a quasigroup $(L, \cdot)$ and that this is enciphered by multiplying by another element y of L so that the encoded message is yx. At the receiving end, the message is deciphered by multiplying by the right inverse y^{ρ} of y. If a left(right) inverse quasigroup is used and the left(right) inverse of x is x^{λ} (x^{ρ}), then the left(right) inverse of x^{λ} (x^{ρ}) is necessarily x. But if a CIPQ is used, this is not necessary the situation. This fact makes an attack on the system more difficult in the case of CIPQs.

Definition 2.4.1 *(Smarandache Keedwell CIPQ)*

Let Q be an initial S-quasigroup with an initial S-subquasigroup P. Q is called a Smarandache Keedwell CIPQ(SKCIPQ) if P is isomorphic to the Keedwell CIPQ, say under a mapping ϕ.

Theorem 2.4.2 *Let $\mathfrak{F}$ be any class of variety of S-quasigroups(loops). Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups(S-loops) that are S-isotopic under the triple of the form*

$$(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1}) \text{ for all } \beta \in SAUM(U) \text{ and some } \delta, \gamma \in SAUM(V)$$

such that their SAGs are non-trivial and are conjugates in $SSYM(L)$ *i.e there exists a* $\psi \in SSYM(L)$ *such that for any*

$$\gamma \in SAUM(V),\ \gamma = \psi^{-1}\alpha\psi \text{ where } \alpha \in SAUM(U).$$

Then, $U \in \mathfrak{F}$ *if and only if* $V \in \mathfrak{F}$.

Proof

By Theorem 2.3.4, $H_S(U) \cong H_S(V)$. Let $U \in \mathfrak{F}$, then since $H(U)$ has an initial S-subquasigroup(S-subloop) that is isomorphic to U and that initial S-subquasigroup(S-subloop) is isomorphic to an S-subquasigroup(S-subloop) of $H(V)$ which is isomorphic to V, $V \in \mathfrak{F}$. The proof for the converse is similar.

Theorem 2.4.3 *Let* $U = (L, \oplus)$ *and* $V = (L, \otimes)$ *be initial S-quasigroups(S-loops) that are S-isotopic under the triple of the form*

$$(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1}) \text{ for all } \beta \in SAUM(U) \text{ and some } \delta, \gamma \in SAUM(V)$$

such that their Smarandache automorphism groups are non-trivial and are conjugates in $SSYM(L)$ *i.e there exists a* $\psi \in SSYM(L)$ *such that for any*

$$\gamma \in SAUM(V),\ \gamma = \psi^{-1}\alpha\psi \text{ where } \alpha \in SAUM(U).$$

Then, U *is a SCIPQ(SCIPL) if and only if* V *is a SCIPQ(SCIPL).*

Proof

Following Theorem 2.3.4, $H_S(U) \gtrsim H_S(V)$. Also, by Theorem 2.3.12, $H_S(U)(H_S(V))$ is a SCIPQ(SCIPL) if and only if $SAUM(U) = \{I\}(SAUM(V) = \{I\})$ and $U(V)$ is a SCIPQ(SCIPL).

Let U be an SCIPQ(SCIPL), then since $H_S(U)$ has a subquasigroup(subloop) that is isomorphic to a S-CIP-subquasigroup(subloop) of U and that subquasigroup(subloop) is isomorphic to a S-subquasigroup(subloop) of $H_S(V)$ which is isomorphic to a S-subquasigroup(subloop) of V, V is a SCIPQ(SCIPL). The proof for the converse is similar.

Application To Cryptography Let the Smarandache Keedwell CIPQ be the SCIPQ U in Theorem 2.4.3. Definitely, its Smarandache automorphism group is non-trivial because as shown in Theorem 2.1 of Keedwell [51], for any CIPQ, the mapping $J_\rho : x \to x^\rho$ is an automorphism. This mapping will be trivial only if the S-CIP-subquasigroup of U is unipotent. For instance, in Example 2.1 of Keedwell [51], the CIPQ $(G, \circ)$ obtained is unipotent because it was constructed using the cyclic group $C_5 =< c : c^5 = e >$ and defined as $a \circ b = a^3b^2$. But in Example 2.2, the CIPQ gotten is not unipotent as a result of using the cyclic group $C_{11} =< c : c^{11} = e >$. Thus, the choice of a Smarandache Keedwell CIPQ which suits our purpose in this work for a cyclic group of order n is one in which $rs = n + 1$ and $r + s \neq n$. Now that we have seen a sample for the choice of U, the initial S-quasigroup V can then be obtained as shown in Theorem 2.4.3. By Theorem 2.4.3, V is a SCIPQ.

Now, according to Theorem 2.3.4, by the choice of the mappings $\alpha, \beta \in SAUM(U)$ and $\psi \in SSYM(L)$ to get the mappings δ, γ, a SCIPQ V can be produced following Theorem 2.4.3. So, the secret keys for the systems are

$$\{\alpha, \beta, \psi, \phi\} \equiv \{\delta, \gamma, \phi\}.$$

Thus whenever a set of information or messages is to be transmitted, the sender will encipher in the Smarandache Keedwell CIPQ by using specifically the S-CIP-subquasigroup in

it(as described earlier on in the introduction) and then enciphere again with

$$\{\alpha, \beta, \psi, \phi\} \equiv \{\delta, \gamma, \phi\}$$

to get a SCIPQ V which is the set of encoded messages. At the receiving end, the message V is deciphered by using an inverse isotopism i.e inverse key of

$$\{\alpha, \beta, \psi\} \equiv \{\delta, \gamma\})$$

to get U and then deciphere again(as described earlier on in the introduction) to get the messages. The secret key can be changed over time. The method described above is a double encryption and its a double protection. It protects each piece of information(element of the quasigroup) and protects the combined information(the quasigroup as a whole). Its like putting on a pair of socks and shoes or putting on under wears and clothes, the body gets better protection. An added advantage of the use of Smarandache Keedwell CIPQ over Keedwell CIPQ in double encryption is that the since the S-CIP-subquasigroups of the Smarandache Keedwell CIPQ in use could be more than one, then, the S-CIP-subquasigroups can be replaced overtime.

Chapter 3

PARASTROPHY OF SMARANDACHE QUASIGROUPS AND LOOPS

3.1 Parastrophy Of Quasigroups And Loops

It has been noted that every quasigroup $(L, \cdot)$ belongs to a set of 6 quasigroups, called adjugates by Fisher and Yates [35], conjugates by Stein [84], [83] and Belousov [17] and parastrophes by Sade [78]. They have been studied by Artzy [10], Charles Lindner and Dwight Steedley [65] and a detailed study on them can be found in [72], [25] and [29]. The most recent studies of the parastrophes of a quasigroup(loop) are by Sokhatskii [81, 82], Duplak [30] and Shchukin and Gushan [80]. For a quasigroup $(L, \cdot)$, its parastrophes are denoted by (L, π_i), $i \in \{1, 2, 3, 4, 5, 6\}$ hence one can take $(L, \cdot) = (L, \pi_1)$. A quasigroup which is equivalent to all its parastrophes is called a totally symmetric quasigroup(introduced by Bruck [20]) while its loop is called a Steiner loop.

Definition 3.1.1 *Let (G, θ) be a quasigroup. The 5 parastrophes or conjugates or adjugates*

of (G,θ) *are quasigroups*

$$(G,\theta^*)\ ,\ (G,\theta^{-1})\ ,\ (G,{}^{-1}\theta)\ ,\ \left(G,\left(\theta^{-1}\right)^*\right)\ \textit{and}\ \left(G,\left({}^{-1}\theta\right)^*\right)$$

whose binary operations

$$\theta^*\ ,\ \theta^{-1}\ ,\ {}^{-1}\theta\ ,\ (\theta^{-1})^*\ \textit{and}\ ({}^{-1}\theta)^*$$

defined on L *satisfies the conditions :*

$$y\theta^* x = z \Leftrightarrow x\theta y = z\ \forall\ x,y,z \in G$$

$$x\theta^{-1} z = y \Leftrightarrow x\theta y = z\ \forall\ x,y,z \in G$$

$$z\ {}^{-1}\theta y = x \Leftrightarrow x\theta y = z\ \forall\ x,y,z \in G$$

$$z\left(\theta^{-1}\right)^* x = y \Leftrightarrow x\theta y = z\ \forall\ x,y,z \in G$$

and

$$y\left({}^{-1}\theta\right)^* z = x \Leftrightarrow x\theta y = z\ \forall\ x,y,z \in G$$

respectively.

Definition 3.1.2 *Let* (G,θ) *be a quasigroup.*

(a) R_x *and* L_x *represent the right and left translation maps in* (G,θ) *for all* $x \in G$.

(b) R_x^* *and* L_x^* *represent the right and left translation maps in* (G,θ^*) *for all* $x \in G$.

(c) $\mathcal{R}_x$ *and* $\mathcal{L}_x$ *represent the right and left translation maps in* (G,θ^{-1}) *for all* $x \in G$.

(d) $\mathbb{R}_x$ *and* $\mathbb{L}_x$ *represent the right and left translation maps in* $(G,{}^{-1}\theta)$ *for all* $x \in G$.

(e) $\mathcal{R}_x^*$ *and* $\mathcal{L}_x^*$ *represent the right and left translation maps in* $(G,(\theta^{-1})^*)$ *for all* $x \in G$.

(f) *$\mathbb{R}_x^*$ and $\mathbb{L}_x^*$ represent the right and left translation maps in $(G, ({}^{-1}\theta)^*)$ for all $x \in G$.*

Remark 3.1.1 *If (L, θ) is a loop, (L, θ^*) is also a loop(and vice versa) while the other adjugates are quasigroups. Furthermore,*

$$(L, \theta^{-1}) \text{ and } (L, ({}^{-1}\theta)^*)$$

have left identity elements, that is they are left loops while

$$(L, {}^{-1}\theta) \text{ and } (L, (\theta^{-1})^*)$$

have right identity elements, that is they are right loops.

$$(L, \theta^{-1}) \text{ or } (L, {}^{-1}\theta) \text{ or } (L, (\theta^{-1})^*) \text{ or } (L, ({}^{-1}\theta)^*)$$

is a loop if and only if (L, θ) is a loop of exponent 2.

Lemma 3.1.1 *If (L, θ) is a quasigroup, then*

1. $R_x^* = L_x \ , \ L_x^* = R_x \ , \ \mathcal{L}_x = L_x^{-1} \ , \ \mathbb{R}_x = R_x^{-1} \ , \ \mathcal{R}_x^* = L_x^{-1} \ , \ \mathbb{L}_x^* = R_x^{-1} \ \forall \ x \in L.$

2. $\mathcal{L}_x = R_x^{*-1} \ , \ \mathbb{R}_x = L_x^{*-1} \ , \ \mathcal{R}_x^* = R_x^{*-1} = \mathcal{L}_x \ , \ \mathbb{L}_x^* = L_x^{*-1} = \mathbb{R}_x \ \forall \ x \in L.$

Proof

The proof of these follows by using Definition 3.1.1 and Definition 3.1.1.

(1)

$$y\theta^* x = z \Leftrightarrow x\theta y = z \Rightarrow y\theta^* x = x\theta y \Rightarrow yR_x^* = yL_x \Rightarrow R_x^* = L_x.$$

$$\text{Also, } y\theta^* x = x\theta y \Rightarrow xL_y^* = xR_y \Rightarrow L_y^* = R_y.$$

$$x\theta^{-1} z = y \Leftrightarrow x\theta y = z \Rightarrow x\theta(x\theta^{-1} z) = z \Rightarrow x\theta z\mathcal{L}_x = z \Rightarrow z\mathcal{L}_x L_x = z \Rightarrow \mathcal{L}_x L_x = I.$$

$$\text{Also, } x\theta^{-1}(x\theta y) = y \Rightarrow x\theta^{-1} yL_x = y \Rightarrow yL_x\mathcal{L}_x = y \Rightarrow L_x\mathcal{L}_x = I.$$

$$\text{Hence, } \mathcal{L}_x = L_x^{-1} \ \forall \ x \in L.$$

$$z({}^{-1}\theta)y = x \Leftrightarrow x\theta y = z \Rightarrow (x\theta y)({}^{-1}\theta)y = x \Rightarrow xR_y({}^{-1}\theta)y = x \Rightarrow xR_y\mathcal{R}_y = x \Rightarrow R_y\mathcal{R}_y = I.$$

$$\text{Also, } (z({}^{-1}\theta)y)\theta y = z \Rightarrow z\mathcal{R}_y\theta y = z \Rightarrow z\mathcal{R}_y R_y = z \Rightarrow \mathcal{R}_y R_y = I.$$

$$\text{Thence, } \mathcal{R}_y = R_y^{-1} \ \forall \ x \in L.$$

$$z(\theta^{-1})^* x = y \Leftrightarrow x\theta y = z, \text{ so, } x\theta(z(\theta^{-1})^* x) = z \Rightarrow x\theta z\mathcal{R}_x^* = z \Rightarrow z\mathcal{R}_x^* L_x = z \Rightarrow \mathcal{R}_x^* L_x = I.$$

$$\text{Also, } (x\theta y)(\theta^{-1})^* x = y \Rightarrow yL_x(\theta^{-1})^* x = y \Rightarrow yL_x\mathcal{R}_x^* = y \Rightarrow L_y\mathcal{R}_x^* = I.$$

$$\text{Whence, } \mathcal{R}_x^* = L_x^{-1}.$$

$$y({}^{-1}\theta)^* z = x \Leftrightarrow x\theta y = z, \text{ so}, y({}^{-1}\theta)^*(x\theta y) = x \Rightarrow y({}^{-1}\theta)^* xR_y = x \Rightarrow xR_y\mathcal{L}_y^* = x \Rightarrow R_y\mathcal{L}_y^* =$$

$$\text{Also, } (y({}^{-1}\theta)^* z)\theta y = z \Rightarrow z\mathcal{L}_y^*\theta y = z \Rightarrow z\mathcal{L}_y^* R_y = z \Rightarrow \mathcal{L}_y^* R_y = I.$$

$$\text{Thus, } \mathcal{L}_y^* = R_y^{-1}.$$

(2) These ones follow from (1).

Lemma 3.1.2 *Every quasigroup which is a Smarandache quasigroup has at least a subgroup.*

Proof

If a quasigroup $(L, \cdot)$ is a SQ, then there exists a subquasigroup $S \subset L$ such that $(S, \cdot)$ is associative. According [60], every quasigroup satisfying the associativity law has an identity hence it is a group. So, S is a subgroup of L.

Theorem 3.1.1 *(Khalil Conditions [79])*

A quasigroup is an isotope of a group if and only if any one of the following six identities are true in the quasigroup for all elements x, y, z, u, v.

1. $x\{z\backslash[(z/u)v]\} = \{[x(z\backslash z)]/u\}v$
2. $x\{u\backslash[(z/u)v]\} = \{[x(u\backslash z)]/u\}v$
3. $x\{z\backslash[(u/u)v]\} = \{[x(z\backslash u)]/u\}v$
4. $x[y\backslash\{[(yy)/z]u\}] = [\{x[y\backslash(yy)]\}/z]u$
5. $x[y\backslash\{[(yz)/y]u\}] = [\{x[y\backslash(yz)]\}/y]u$
6. $x[z\backslash\{[(yy)/y]u\}] = [\{x[z\backslash(yy)]\}/y]u$

3.2 Parastrophy Of Smarandache Quasigroups

Theorem 3.2.1 (L, θ) *is a Smarandache quasigroup with associative subquasigroup* (S, θ) *if and only if any of the following equivalent statements is true.*

1. (S, θ) *is isotopic to* $(S, (\theta^{-1})^*)$.
2. (S, θ^*) *is isotopic to* (S, θ^{-1}).
3. (S, θ) *is isotopic to* $(S, ({}^{-1}\theta)^*)$.
4. (S, θ^*) *is isotopic to* $(S, {}^{-1}\theta)$.

Proof

L is a SQ with associative subquasigroup S if and only if

$$s_1\theta(s_2\theta s_3) = (s_1\theta s_2)\theta s_3 \Leftrightarrow$$

$$R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow L_{s_1\theta s_2} = L_{s_2}L_{s_1} \ \forall \ s_1, s_2, s_3 \in S.$$

The proof of the equivalence of (1) and (2) is as follows.

$$L_{s_1\theta s_2} = L_{s_2}L_{s_1} \Leftrightarrow \mathcal{L}^{-1}_{s_1\theta s_2} = \mathcal{L}^{-1}_{s_2}\mathcal{L}^{-1}_{s_1} \Leftrightarrow$$

$$\mathcal{L}_{s_1\theta s_2} = \mathcal{L}_{s_1}\mathcal{L}_{s_2} \Leftrightarrow (s_1\theta s_2)\theta^{-1}s_3 = s_2\theta^{-1}(s_1\theta^{-1}s_3) \Leftrightarrow$$

$$(s_1\theta s_2)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow$$

$$(s_2\theta^* s_1)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} \Leftrightarrow (\mathcal{R}_{s_3}, I, \mathcal{R}_{s_3}) \ : \ (S,\theta) \to (S,(\theta^{-1})^*) \Leftrightarrow$$

$$(I, \mathcal{R}_{s_3}, \mathcal{R}_{s_3}) \ : \ (S,\theta^*) \to (S,\theta^{-1}) \Leftrightarrow (S,\theta)$$

is isotopic to $(S,(\theta^{-1})^*) \Leftrightarrow (S,\theta^*)$ is isotopic to (S,θ^{-1}).

The proof of the equivalence of (3) and (4) is as follows.

$$R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow I\!R^{-1}_{s_2}I\!R^{-1}_{s_3} = I\!R^{-1}_{s_2\theta s_3} \Leftrightarrow$$

$$I\!R_{s_3}I\!R_{s_2} = I\!R_{s_2\theta s_3} \Leftrightarrow ({s_1}^{-1}\theta s_3)^{-1}\theta s_2 = {s_1}^{-1}\theta(s_2\theta s_3) \Leftrightarrow$$

$$(s_2\theta s_3)I\!L_{s_1} = s_3I\!L_{{s_1}^{-1}\theta s_2} = s_2({}^{-1}\theta)^* s_3 I\!L_{s_1} \Leftrightarrow$$

$$(s_2\theta s_3)I\!L_{s_1} = s_2({}^{-1}\theta)^* s_3 I\!L_{s_1} \Leftrightarrow (s_3\theta^* s_2)I\!L_{s_1} = s_3I\!L_{{s_1}^{-1}\theta s_2} \Leftrightarrow$$

$$(I, I\!L_{s_1}, I\!L_{s_1}) \ : \ (S,\theta) \to (S,({}^{-1}\theta)^*) \Leftrightarrow (I\!L_{s_1}, I, I\!L_{s_1}) \ : \ (S,\theta^*) \to (S,{}^{-1}\theta) \Leftrightarrow (S,\theta)$$

is isotopic to $(S,({}^{-1}\theta)^*) \Leftrightarrow (S,\theta^*)$ is isotopic to $(S,{}^{-1}\theta)$.

Remark 3.2.1 *In the proof of Theorem 3.2.1, it can be observed that the isotopisms are triples of the forms (A,I,A) and (I,B,B). All weak associative identities such as the Bol, Moufang and extra identities have been found to be isotopic invariant in loops for any triple of the form (A,B,C) while the central identities have been found to be isotopic invariant only under triples of the forms (A,B,A) and (A,B,B). Since associativity obeys all the Bol-Moufang identities, the observation in the theorem agrees with the latter stated facts.*

Corollary 3.2.1 *(L, θ) is a Smarandache quasigroup with associative subquasigroup (S, θ) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof

Let (L, θ) be the quasigroup in consideration. By Lemma 3.1.2, (S, θ) is a group. Notice that

$$R_{s_2} R_{s_3} = R_{s_2 \theta s_3} \Leftrightarrow L^*_{s_2 \theta s_3} = L^*_{s_3} L^*_{s_2}.$$

Hence, (S, θ^*) is also a group. In Theorem 3.2.1, two of the parastrophes are isotopes of (S, θ) while the other two are isotopes of (S, θ^*). Since the Khalil conditions are neccessary and sufficient conditions for a quasigroup to be an isotope of a group, then they must be necessarily and sufficiently true in the four quasigroup parastrophes of (S, θ).

Lemma 3.2.1 *(L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the following equivalent statements is true.*

1. *(S, θ^*) is isotopic to $(S, {}^{-1}\theta)$.*
2. *(S, θ) is isotopic to $(S, ({}^{-1}\theta)^*)$.*
3. *(S, θ^*) is isotopic to (S, θ^{-1}).*
4. *(S, θ) is isotopic to $(S, (\theta^{-1})^*)$.*

Proof

Replace (L, θ) with (L, θ^*) in Theorem 3.2.1.

Corollary 3.2.2 *(L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof

Replace (L, θ) with (L, θ^*) in Corollary 3.2.1.

Lemma 3.2.2 (L, θ^{-1}) *is a Smarandache quasigroup with associative subquasigroup* (S, θ^{-1}) *if and only if any of the following equivalent statements is true.*

1. (S, θ^{-1}) *is isotopic to* (S, θ^{*}) .
2. $(S, (\theta^{-1})^{*})$ *is isotopic to* (S, θ).
3. (S, θ^{-1}) *is isotopic to* $(S, {}^{-1}\theta)$.
4. $(S, (\theta^{-1})^{*})$ *is isotopic to* $(S, ({}^{-1}\theta)^{*})$.

Proof

Replace (L, θ) with (L, θ^{-1}) in Theorem 3.2.1.

Corollary 3.2.3 (L, θ^{-1}) *is a Smarandache quasigroup with associative subquasigroup* (S, θ^{-1}) *if and only if any of the six Khalil conditions is true for some four parastrophes of* (S, θ).

Proof

Replace (L, θ) with (L, θ^{-1}) in Corollary 3.2.1.

Lemma 3.2.3 $(L, {}^{-1}\theta)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, {}^{-1}\theta)$ *if and only if any of the following equivalent statements is true.*

1. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^{-1}).
2. $(S, ({}^{-1}\theta)^{*})$ *is isotopic to* $(S, (\theta^{-1})^{*})$.
3. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^{*}).
4. $(S, ({}^{-1}\theta)^{*})$ *is isotopic to* (S, θ).

Proof

Replace (L, θ) with $(L, {}^{-1}\theta)$ in Theorem 3.2.1.

Corollary 3.2.4 $(L, {}^{-1}\theta)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, {}^{-1}\theta)$ *if and only if any of the six Khalil conditions is true for some four parastrophes of* (S, θ).

Proof

Replace (L, θ) with $(L, {}^{-1}\theta)$ in Corollary 3.2.1.

Lemma 3.2.4 $(L, (\theta^{-1})^*)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, (\theta^{-1})^*)$ *if and only if any of the following equivalent statements is true.*

1. $(S, (\theta^{-1})^*)$ *is isotopic to* $(S, ({}^{-1}\theta)^*)$.
2. (S, θ^{-1}) *is isotopic to* $(S, {}^{-1}\theta)$.
3. $(S, (\theta^{-1})^*)$ *is isotopic to* (S, θ).
4. $(S, \theta^{-1}))$ *is isotopic to* (S, θ^*).

Proof

Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Theorem 3.2.1.

Corollary 3.2.5 $(L, (\theta^{-1})^*)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, (\theta^{-1})^*)$ *if and only if any of the six Khalil conditions is true for some four parastrophes of* (S, θ).

Proof

Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Corollary 3.2.1.

Lemma 3.2.5 $(L, ({}^{-1}\theta)^*)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, ({}^{-1}\theta)^*)$ *if and only if any of the following equivalent statements is true.*

1. $(S, ({}^{-1}\theta)^*)$ *is isotopic to* (S, θ).

2. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^*).

3. $(S, ({}^{-1}\theta)^*)$ *is isotopic to* $(S, (\theta^{-1})^*)$.

4. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^{-1}).

Proof

Replace (L, θ) with $(L, ({}^{-1}\theta)^*)$ in Theorem 3.2.1.

Corollary 3.2.6 $(L, ({}^{-1}\theta)^*)$ *is a Smarandache quasigroup with associative subquasigroup* $(S, ({}^{-1}\theta)^*)$ *if and only if any of the six Khalil conditions is true for some four parastrophes of* (S, θ).

Proof

Replace (L, θ) with $(L, ({}^{-1}\theta)^*)$ in Corollary 3.2.1.

Theorem 3.2.2 (L, π_i) *is a Smarandache quasigroup with associative subquasigroup* (S, π_i) $\forall$ $i \in \{1, 2, 3, 4, 5, 6\}$ *if and only if for any of some four* $j \in \{1, 2, 3, 4, 5, 6\}$, (S, π_j) *is an isotope of* (S, π_i) *or* (S, π_k) *for one* $k \in \{1, 2, 3, 4, 5, 6\}$ *such that* $i \neq j \neq k$.

Proof

This is simply the summary of Theorem 3.2.1, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5.

Corollary 3.2.7 (L, π_i) *is a Smarandache quasigroup with associative subquasigroup* (S, π_i) $\forall$ $i \in \{1, 2, 3, 4, 5, 6\}$ *if and only if any of the six Khalil conditions is true for any of some four of* (S, π_i).

Proof

This can be deduced from Theorem 3.2.2 and the Khalil conditions or by combining Corollary 3.2.1, Corollary 3.2.2, Corollary 3.2.3, Corollary 3.2.4, Corollary 3.2.5 and Corollary 3.2.6.

Chapter 4

UNIVERSALITY OF SMARANDACHE LOOPS OF BOL-MOUFANG TYPE

4.1 Smarandache f, g-Principal Isotopes

A subloop(subquasigroup) $(S, \otimes)$ of a loop(quasigroup) $(G, \otimes)$ is called a Smarandache f, g-principal isotope of the subloop(subquasigroup) $(S, \oplus)$ of a loop(quasigroup) $(G, \oplus)$ if for some $f, g \in S$,

$$xR_g \otimes yL_f = (x \oplus y) \ \forall \ x, y \in S.$$

On the other hand $(G, \otimes)$ is called a Smarandache f, g-principal isotope of $(G, \oplus)$ if for some $f, g \in S$,

$$xR_g \otimes yL_f = (x \oplus y) \ \forall \ x, y \in G$$

where $(S, \oplus)$ is a S-subquasigroup(S-subloop) of $(G, \oplus)$. In these cases, f and g are called Smarandache elements(S-elements).

Theorem 4.1.1 *([21]) Let $(G, \oplus)$ and $(H, \otimes)$ be two distinct isotopic loops(quasigroups). There exists an f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$.*

Corollary 4.1.1 *Let $\mathcal{P}$ be an isotopic invariant property in loops(quasigroups). If $(G, \oplus)$ is a loop(quasigroup) with the property $\mathcal{P}$, then $(G, \oplus)$ is a universal loop(quasigroup) relative to the property $\mathcal{P}$ if and only if every f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ has the property $\mathcal{P}$.*

Proof

If $(G, \oplus)$ is a universal loop relative to the property $\mathcal{P}$ then every distinct loop isotope $(H, \otimes)$ of $(G, \oplus)$ has the property $\mathcal{P}$. By Theorem 4.1.1, there exists an f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$. Hence, since $\mathcal{P}$ is an isomorphic invariant property, every $(G, \circ)$ has it.

Conversely, if every f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ has the property $\mathcal{P}$ and since by Theorem 4.1.1 for each distinct isotope $(H, \otimes)$ there exists an f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$, then all $(H, \otimes)$ has the property. Thus, $(G, \oplus)$ is a universal loop relative to the property $\mathcal{P}$.

Lemma 4.1.1 *Let $(G, \oplus)$ be a loop(quasigroup) with a subloop(subquasigroup) $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then $(S, \circ)$ is a subloop(subquasigroup) of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$.*

Proof

If $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$, then for some $f, g \in S$,

$$xR_g \circ yL_f = (x \oplus y) \ \forall \ x, y \in S \Rightarrow x \circ y = xR_g^{-1} \oplus yL_f^{-1} \in S \ \forall \ x, y \in S$$

since $f, g \in S$. So, $(S, \circ)$ is a subgroupoid of $(G, \circ)$. $(S, \circ)$ is a subquasigroup follows from the fact that $(S, \oplus)$ is a subquasigroup. $f \oplus g$ is a two sided identity element in $(S, \circ)$. Thus,

$(S, \circ)$ is a subloop of $(G, \circ)$.

4.2 Universality Of Smarandache Loops

Theorem 4.2.1 *A Smarandache quasigroup is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes.*

Proof

Let $(G, \oplus)$ be a Smarandache quasigroup with a S-subquasigroup $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subquasigroup of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

It shall now be shown that

$$(x \circ y) \circ z = x \circ (y \circ z) \ \forall \ x, y, z \in S.$$

But in the quasigroup $(G, \oplus)$, xy will have preference over $x \oplus y \ \forall \ x, y \in G$.

$$(x \circ y) \circ z = (xR_g^{-1} \oplus yL_f^{-1}) \circ z = (xg^{-1} \oplus f^{-1}y) \circ z = (xg^{-1} \oplus f^{-1}y)R_g^{-1} \oplus zL_f^{-1}$$

$$= (xg^{-1} \oplus f^{-1}y)g^{-1} \oplus f^{-1}z = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.$$

$$x \circ (y \circ z) = x \circ (yR_g^{-1} \oplus zL_f^{-1}) = x \circ (yg^{-1} \oplus f^{-1}z) = xR_g^{-1} \oplus (yg^{-1} \oplus f^{-1}z)L_f^{-1}$$

$$= xg^{-1} \oplus f^{-1}(yg^{-1} \oplus f^{-1}z) = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.$$

Thus, $(S, \circ)$ is an S-subquasigroup of $(G, \circ)$ hence, $(G, \circ)$ is a S-quasigroup. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$.

So we can now choose the isomorphic image of $(S, \circ)$ which will now be an S-subquasigroup in $(H, \otimes)$. So, $(H, \otimes)$ is an S-quasigroup. This conclusion can also be drawn straight from Corollary 4.1.1.

Theorem 4.2.2 *A Smarandache loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. But if a Smarandache loop is universal then*

$$(I, L_f R_g^{-1} R_{f\rho} L_f^{-1}, R_g^{-1} R_{f\rho})$$

is an autotopism of an S-subloop of the S-loop such that f and g are S-elements.

Proof

Every loop is a quasigroup. Hence, the first claim follows from Theorem 4.2.1. The proof of the second part is as follows. If a Smarandache loop $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is an S-loop i.e there exists an S-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 4.1.1, $(G, \circ)$ is an S-loop with say an S-subloop $(S, \circ)$. So,

$$(x \circ y) \circ z = x \circ (y \circ z) \ \forall \ x, y, z \in S$$

where

$$x \circ y = x R_g^{-1} \oplus y L_f^{-1} \ \forall \ x, y \in S.$$

$$(x R_g^{-1} \oplus y L_f^{-1}) R_g^{-1} \oplus z L_f^{-1} = x R_g^{-1} \oplus (y R_g^{-1} \oplus z L_f^{-1}) L_f^{-1}.$$

Replacing $x R_g^{-1}$ by x', $y L_f^{-1}$ by y' and taking $z = e$ in $(S, \oplus)$ we have;

$$(x' \oplus y') R_g^{-1} R_{f\rho} = x' \oplus y' L_f R_g^{-1} R_{f\rho} L_f^{-1} \Rightarrow (I, L_f R_g^{-1} R_{f\rho} L_f^{-1}, R_g^{-1} R_{f\rho})$$

is an autotopism of an S-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

4.3 Universality of Smarandache Bol Loops, Moufang Loops and Extra Loops

Theorem 4.3.1 *A Smarandache right(left)Bol loop is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes. But, if a Smarandache right(left)Bol loop is universal then*

$$\mathcal{T}_1 = (R_g R_{f^\rho}^{-1}, L_{g^\lambda} R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho}) \Bigg(\mathcal{T}_2 = (R_{f^\rho} L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f L_{g^\lambda}^{-1}, L_f^{-1} L_{g^\lambda}) \Bigg)$$

is an autotopism of an SRB(SLB)-subloop of the SRBL(SLBL) such that f and g are S-elements.

Proof

Let $(G,\oplus)$ be a SRBL(SLBL) with a S-RB(LB)-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 4.1.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x \circ y = x R_g^{-1} \oplus y L_f^{-1} \ \forall \ x, y \in S.$$

It is already known from [72] that RB(LB) loops are universal, hence $(S,\circ)$ is a RB(LB) loop thus an S-RB(LB)-subloop of $(G,\circ)$. By Theorem 4.1.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes) \cong (G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ which will now be an S-RB(LB)-subloop in $(H,\otimes)$. So, $(H,\otimes)$ is an SRBL(SLBL). This conclusion can also be drawn straight from Corollary 4.1.1.

The proof of the second is as follows. If a SRBL(SLBL) $(G,\oplus)$ is universal then every isotope $(H,\otimes)$ is an SRBL(SLBL) i.e there exists an S-RB(LB)-subloop $(S,\otimes)$ in $(H,\otimes)$. Let $(G,\circ)$ be the f,g-principal isotope of $(G,\oplus)$, then by Corollary 4.1.1, $(G,\circ)$ is an

SRBL(SLBL) with say an SRB(SLB)-subloop $(S, \circ)$. So for an SRB-subloop $(S, \circ)$,

$$[(y \circ x) \circ z] \circ x = y \circ [(x \circ z) \circ x] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]R_g^{-1} \oplus xL_f^{-1} = yR_g^{-1} \oplus [(xR_g^{-1} \oplus zL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1} \oplus z')R_g^{-1}R_{f^\rho} = y' \oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$(y'' \oplus z')R_g^{-1}R_{f^\rho} = y''R_gR_{f^\rho}^{-1} \oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1} \Rightarrow (R_gR_{f^\rho}^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

is an autotopism of an SRB-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

On the other hand, for a SLB-subloop $(S, \circ)$,

$$[x \circ (y \circ x)] \circ z = x \circ [y \circ (x \circ z)] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[xR_g^{-1} \oplus (yR_g^{-1} \oplus xL_f^{-1})L_f^{-1}]R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z' = (y' \oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda} \Rightarrow (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SLB-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Theorem 4.3.2 *A Smarandache Moufang loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. But, if a Smarandache Moufang loop is universal then*

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}), (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}), (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}), (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

are autotopisms of an SM-subloop of the SML such that f and g are S-elements.

Proof

Let $(G, \oplus)$ be a SML with a SM-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-

principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

It is already known from [72] that Moufang loops are universal, hence $(S, \circ)$ is a Moufang loop thus an SM-subloop of $(G, \circ)$. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be an SM-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is an SML. This conclusion can also be drawn straight from Corollary 4.1.1.

The proof of the second part is as follows. If a SML $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is an SML i.e there exists an SM-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 4.1.1, $(G, \circ)$ is an SML with say an SM-subloop $(S, \circ)$. For an SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = [x \circ (y \circ z)] \circ x \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = [xR_g^{-1} \oplus (yR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]R_g^{-1} \oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Again, for an SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = x \circ [(y \circ z) \circ x] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = xR_g^{-1} \oplus [(yR_g^{-1} \oplus zL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Also, if $(S, \circ)$ is an SM-subloop then,

$$[(x \circ y) \circ x] \circ z = x \circ [y \circ (x \circ z)] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1} \oplus z' = (y' \oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda} \Rightarrow (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Furthermore, if $(S, \circ)$ is an SM-subloop then,

$$[(y \circ x) \circ z] \circ x = y \circ [x \circ (z \circ x)] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]R_g^{-1} \oplus xL_f^{-1} = yR_g^{-1} \oplus [xR_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1} \oplus z')R_g^{-1}R_{f^\rho} = y' \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$(y''\oplus z')R_g^{-1}R_{f^\rho} = y''R_gR_{f^\rho}^{-1}\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1} \Rightarrow (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

is an autotopism of an SM-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Lastly, $(S,\oplus)$ is an SM-subloop if and only if $(S,\circ)$ is an SRB-subloop and an SLB-subloop. So by Theorem 4.3.1, $\mathcal{T}_1$ and $\mathcal{T}_2$ are autotopisms in $(S,\oplus)$, hence $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_2\mathcal{T}_1$ are autotopisms in $(S,\oplus)$.

Theorem 4.3.3 *A Smarandache extra loop is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes. But, if a Smarandache extra loop is universal then* $(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_{f^\rho}^{-1}R_gL_f^{-1}, L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g)$,

$$(R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}L_f^{-1}, L_f^{-1}L_{g^\lambda}), (R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}), (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}), (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}), (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

are autotopisms of an SE-subloop of the SEL such that f and g are S-elements.

Proof

Let $(G,\oplus)$ be a SEL with a SE-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 4.1.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

In [39] and [62] respectively, it was shown and stated that a loop is an extra loop if and only if it is a Moufang loop and a CC-loop. But since CC-loops are G-loops(they are isomorphic to all loop isotopes) then extra loops are universal, hence $(S, \circ)$ is an extra loop thus an SE-subloop of $(G, \circ)$. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be an SE-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is an SEL. This conclusion can also be drawn straight from Corollary 4.1.1. The proof of the second part is as follows. If a SEL $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is an SEL i.e there exists an SE-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 4.1.1, $(G, \circ)$ is an SEL with say an SE-subloop $(S, \circ)$. For an SE-subloop $(S, \circ)$,

$$[(x \circ y) \circ z] \circ x = x \circ [y \circ (z \circ x)] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]R_g^{-1} \oplus xL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$(y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z')R_g^{-1}R_{f^\rho} = (y' \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}$ by z'' so that

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z''L_fR_{f^\rho}^{-1}R_gL_f^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g \Rightarrow$$

$$(R_g L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f R_{f^\rho}^{-1} R_g L_f^{-1}, L_f^{-1} L_{g^\lambda} R_{f^\rho}^{-1} R_g)$$

is an autotopism of an SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Again, for an SE-subloop $(S, \circ)$,

$$(x \circ y) \circ (x \circ z) = x \circ [(y \circ x) \circ z] \ \forall \ x, y, z \in S$$

where

$$x \circ y = x R_g^{-1} \oplus y L_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(x R_g^{-1} \oplus y L_f^{-1}) R_g^{-1} \oplus (x R_g^{-1} \oplus z L_f^{-1}) L_f^{-1} = x R_g^{-1} \oplus [(y R_g^{-1} \oplus x L_f^{-1}) R_g^{-1} \oplus z L_f^{-1}] L_f^{-1}.$$

Replacing $y R_g^{-1}$ by y', $z L_f^{-1}$ by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y' R_g L_f^{-1} L_{g^\lambda} R_g^{-1} \oplus z' L_{g^\lambda} L_f^{-1} = (y' R_{f^\rho} R_g^{-1} \oplus z') L_f^{-1} L_{g^\lambda}.$$

Again, replace $y' R_{f^\rho} R_g^{-1}$ by y'' so that

$$y'' R_g R_{f^\rho}^{-1} R_g L_f^{-1} L_{g^\lambda} R_g^{-1} \oplus z' L_{g^\lambda} L_f^{-1} = (y'' \oplus z') L_f^{-1} L_{g^\lambda} \Rightarrow (R_g R_{f^\rho}^{-1} R_g L_f^{-1} L_{g^\lambda} R_g^{-1}, L_{g^\lambda} L_f^{-1}, L_f^{-1} L_{g^\lambda})$$

is an autotopism of an SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Also, if $(S, \circ)$ is an SE-subloop then,

$$(y \circ x) \circ (z \circ x) = [y \circ (x \circ z)] \circ x \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = [(yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]R_g^{-1} \oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_{f^\rho}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z'L_{g^\lambda}L_f^{-1})R_g^{-1}R_{f^\rho}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_{f^\rho}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')R_g^{-1}R_{f^\rho} \Rightarrow (R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

is an autotopism of an SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Lastly, $(S, \oplus)$ is an SE-subloop if and only if $(S, \circ)$ is an SM-subloop and an SCC-subloop. So by Theorem 4.3.2, the six remaining triples are autotopisms in $(S, \oplus)$.

4.4 Universality Of Smarandache Inverse Property Loops

Theorem 4.4.1 *A Smarandache left(right) inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache left(right) Bol loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes.*

Proof

Let $(G, \oplus)$ be a SLIPL with a SLIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

$(G, \oplus)$ is a universal SLIPL if and only if every isotope $(H, \otimes)$ is a SLIPL. $(H, \otimes)$ is a SLIPL if and only if it has at least a SLIP-subloop $(S, \otimes)$. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SLIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SLIP-subloop in $(G, \circ)$. As shown in [72], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SLIP-subloops if and only if $(S, \oplus)$ is a left Bol subloop(i.e a SLB-subloop). So, $(G, \oplus)$ is SLBL.

Conversely, if $(G, \oplus)$ is SLBL, then there exists a SLB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SLB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 4.3.1. So, $(S, \circ)$ is a SLB-subloop in $(G, \circ)$. Left Bol loops have the left inverse property(LIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SLIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SLBLs. Therefore, $(G, \oplus)$ is a universal SLIPL.

The proof for a Smarandache right inverse property loop is similar and is as follows. Let

$(G, \oplus)$ be a SRIPL with a SRIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

$(G, \oplus)$ is a universal SRIPL if and only if every isotope $(H, \otimes)$ is a SRIPL. $(H, \otimes)$ is a SRIPL if and only if it has at least a SRIP-subloop $(S, \otimes)$. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SRIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SRIP-subloop in $(G, \circ)$. As shown in [72], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SRIP-subloops if and only if $(S, \oplus)$ is a right Bol subloop(i.e a SRB-subloop). So, $(G, \oplus)$ is SRBL.

Conversely, if $(G, \oplus)$ is SRBL, then there exists a SRB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SRB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 4.3.1. So, $(S, \circ)$ is a SRB-subloop in $(G, \circ)$. Right Bol loops have the right inverse property(RIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SRIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SRBLs. Therefore, $(G, \oplus)$ is a universal SRIPL.

Theorem 4.4.2 *A Smarandache inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache*

Moufang loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes.

Proof

Let $(G, \oplus)$ be a SIPL with a SIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

$(G, \oplus)$ is a universal SIPL if and only if every isotope $(H, \otimes)$ is a SIPL. $(H, \otimes)$ is a SIPL if and only if it has at least a SIP-subloop $(S, \otimes)$. By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SIP-subloop in $(G, \circ)$. As shown in [72], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SIP-subloops if and only if $(S, \oplus)$ is a Moufang subloop(i.e a SM-subloop). So, $(G, \oplus)$ is SML.

Conversely, if $(G, \oplus)$ is SML, then there exists a SM-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 4.1.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

By Theorem 4.1.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SM-subloop in $(H, \otimes)$ using the same reasoning in Theorem 4.3.1. So, $(S, \circ)$ is a SM-subloop in $(G, \circ)$. Moufang loops have the inverse property(IP), hence, $(S, \oplus)$ and $(S, \circ)$ are SIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SMLs. Therefore, $(G, \oplus)$ is a universal SIPL.

Corollary 4.4.1 *If a Smarandache left(right) inverse property loop is universal then*

$$(R_g R_{f^\rho}^{-1}, L_{g^\lambda} R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho})\left((R_{f^\rho} L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f L_{g^\lambda}^{-1}, L_f^{-1} L_{g^\lambda})\right)$$

is an autotopism of an SLIP(SRIP)-subloop of the SLIPL(SRIPL) such that f and g are S-elements.

Proof

This follows by Theorem 4.4.1 and Theorem 4.4.1.

Corollary 4.4.2 *If a Smarandache inverse property loop is universal then*

$$(R_g L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, L_f^{-1} L_{g^\lambda} R_g^{-1} R_{f^\rho}), (R_g L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho} L_f^{-1} L_{g^\lambda}),$$

$$(R_g L_f^{-1} L_{g^\lambda} R_g^{-1} R_{f^\rho} R_g^{-1}, L_f L_{g^\lambda}^{-1}, L_f^{-1} L_{g^\lambda}), (R_g R_{f^\rho}^{-1}, L_f R_g^{-1} R_{f^\rho} L_f^{-1} L_{g^\lambda} L_f^{-1}, R_g^{-1} R_{f^\rho}),$$

$$(R_g L_f^{-1} L_{g^\lambda} R_g^{-1}, L_{g^\lambda} R_g^{-1} R_{f^\rho} L_{g^\lambda}^{-1}, R_g^{-1} R_{f^\rho} L_f^{-1} L_{g^\lambda}), (R_{f^\rho} L_f^{-1} L_{g^\lambda} R_{f^\rho}^{-1}, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, L_f^{-1} L_{g^\lambda} R_g^{-1} R_{f^\rho})$$

are autotopisms of an SIP-subloop of the SIPL such that f and g are S-elements.

Proof

This follows from Theorem 4.4.2 and Theorem 4.3.2.

Chapter 5

SMARANDACHE ISOTOPY OF SMARANDACHE QUASIGROUPS AND LOOPS

Recall that if $(L, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $(G')A = L'$, where $A \in \{U, V, W\}$, then the isotopism

$$(U, V, W) : (L, \cdot) \to (G, \circ)$$

is called a Smarandache isotopism(S-isotopism). Consequently, if $W = I$ the triple (U, V, I) is called a Smarandache principal isotopism. But if in addition G is a S-quasigroup with S-subgroup H' such that for some $f, g \in H$, $U = R_g$ and $V = L_f$, and

$$(R_g, L_f, I) \ : \ (G, \cdot) \to (G, \circ)$$

is an isotopism, then the triple is called a Smarandache f, g-principal isotopism while f and g are called Smarandache elements(S-elements).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L, \cdot) \succsim (G, \circ)$. An S-loop $(L, \cdot)$ is called a G-Smarandache loop(GS-loop) if and only if $(L, \cdot) \succsim (G, \circ)$ for all loop isotopes(or particularly all S-loop isotopes) $(G, \circ)$ of $(L, \cdot)$.

5.1 Smarandache Isotopy And Isomorphy Classes

Theorem 5.1.1 *Let* $\mathfrak{G} = \left\{ \left(G_\omega, \circ_\omega\right) \right\}_{\omega \in \Omega}$ *be a set of distinct S-groupoids with a corresponding set of S-subsemigroups* $\mathfrak{H} = \left\{ \left(H_\omega, \circ_\omega\right) \right\}_{\omega \in \Omega}$. *Define a relation* $\sim$ *on* $\mathfrak{G}$ *such that for all* $\left(G_{\omega_i}, \circ_{\omega_i}\right)$, $\left(G_{\omega_j}, \circ_{\omega_j}\right) \in \mathfrak{G}$, *where* $\omega_i, \omega_j \in \Omega$,

$$\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_j}, \circ_{\omega_j}\right) \Longleftrightarrow \left(G_{\omega_i}, \circ_{\omega_i}\right) \text{ and } \left(G_{\omega_j}, \circ_{\omega_j}\right) \text{ are S-isotopic.}$$

Then $\sim$ *is an equivalence relation on* $\mathfrak{G}$.

Proof

Let

$$\left(G_{\omega_i}, \circ_{\omega_i}\right) , \left(G_{\omega_j}, \circ_{\omega_j}\right), \ \left(G_{\omega_k}, \circ_{\omega_k}\right) \in \mathfrak{G}, \text{ where } \omega_i, \omega_j, \omega_k \in \Omega.$$

Reflexivity If $I \ : \ G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \ \forall \ x, y \in G_{\omega_i} \Longrightarrow \text{ the triple } (I, I, I) \ : \ \left(G_{\omega_i}, \circ_{\omega_i}\right) \to \left(G_{\omega_i}, \circ_{\omega_i}\right)$$

is an S-isotopism since

$$\left(H_{\omega_i}\right)I = H_{\omega_i} \ \forall \ \omega_i \in \Omega.$$

In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

Symmetry Let

$$\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_j}, \circ_{\omega_j}\right).$$

Then there exist bijections

$$U, V, W \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big)$$

is an isotopism. Since each of U, V, W is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

are bijective. In fact,

$$\big(H_{\omega_j}\big)A^{-1} = H_{\omega_i} \ \forall \ A \in \{U, V, W\}$$

since A is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

is an isotopism. Thus,

$$\big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_i}, \circ_{\omega_i}\big).$$

Transitivity Let

$$\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and } \big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big).$$

Then there exist bijections

$$U_1, V_1, W_1 \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and } U_2, V_2, W_2 \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

$$\text{such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U_1, V_1, W_1\}$$

and $\big(H_{\omega_j}\big)B = H_{\omega_k}\ \forall\ B \in \{U_2, V_2, W_2\}$ so that the triples

$$\alpha_1 = (U_1, V_1, W_1)\ :\ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and}$$

$$\alpha_2 = (U_2, V_2, W_2)\ :\ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are isotopisms. Since each of U_i, V_i, W_i, $i = 1, 2$, is bijective, then

$$U_3 = U_1U_2, V_3 = V_1V_2, W_3 = W_1W_2\ :\ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are bijections such that

$$\big(H_{\omega_i}\big)A_3 = \big(H_{\omega_i}\big)A_1A_2 = \big(H_{\omega_j}\big)A_2 = H_{\omega_k}$$

so that the triple

$$\alpha_3 = \alpha_1\alpha_2 = (U_3, V_3, W_3)\ :\ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

is an isotopism. Thus,

$$\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big).$$

Remark 5.1.1 *As a follow up to Theorem 5.1.1, the elements of the set $\mathfrak{G}/\sim$ will be referred to as Smarandache isotopy classes(S-isotopy classes). Similarly, if $\sim$ meant "S-isomorphism" in Theorem 5.1.1, then the elements of $\mathfrak{G}/\sim$ will be referred to as Smarandache isomorphy classes(S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.*

Corollary 5.1.1 *Let $\mathcal{L}_n$, $\mathcal{SL}_n$ and $\mathcal{NSL}_n$ be the sets of; all finite loops of order n; all finite S-loops of order n and all finite non S-loops of order n respectively.*

1. *If $\mathcal{A}_i^n$ and $\mathcal{B}_i^n$ represent the isomorphy class of $\mathcal{L}_n$ and the S-isomorphy class of $\mathcal{SL}_n$ respectively, then*

   (a) $|\mathcal{SL}_n| + |\mathcal{NSL}_n| = |\mathcal{L}_n|$;

   (i) $|\mathcal{SL}_5| + |\mathcal{NSL}_5| = 56$,

   (ii) $|\mathcal{SL}_6| + |\mathcal{NSL}_6| = 9,408$ *and*

   (iii) $|\mathcal{SL}_7| + |\mathcal{NSL}_7| = 16,942,080$.

   (b) $|\mathcal{NSL}_n| = \sum_{i=1} |\mathcal{A}_i^n| - \sum_{i=1} |\mathcal{B}_i^n|$;

   (i) $|\mathcal{NSL}_5| = \sum_{i=1}^{6} |\mathcal{A}_i^5| - \sum_{i=1} |\mathcal{B}_i^5|$,

   (ii) $|\mathcal{NSL}_6| = \sum_{i=1}^{109} |\mathcal{A}_i^6| - \sum_{i=1} |\mathcal{B}_i^6|$ *and*

   (iii) $|\mathcal{NSL}_7| = \sum_{i=1}^{23,746} |\mathcal{A}_i^7| - \sum_{i=1} |\mathcal{B}_i^7|$.

2. *If $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ represent the isotopy class of $\mathcal{L}_n$ and the S-isotopy class of $\mathcal{SL}_n$ respectively, then*

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

   (i) $|\mathcal{NSL}_5| = \sum_{i=1}^{2} |\mathfrak{A}_i^5| - \sum_{i=1} |\mathfrak{B}_i^5|$,

   (ii) $|\mathcal{NSL}_6| = \sum_{i=1}^{22} |\mathfrak{A}_i^6| - \sum_{i=1} |\mathfrak{B}_i^6|$ *and*

   (iii) $|\mathcal{NSL}_7| = \sum_{i=1}^{564} |\mathfrak{A}_i^7| - \sum_{i=1} |\mathfrak{B}_i^7|$.

Proof

An S-loop is an S-groupoid. Thus by Theorem 5.1.1, we have S-isomorphy classes and S-isotopy classes. Recall that

$$|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \bigcap \mathcal{NSL}_n|$$

but

$$\mathcal{SL}_n \bigcap \mathcal{NSL}_n = \emptyset \text{ so } |\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|.$$

As stated and shown in [72], [37], [24] and [66], the facts in Table 5.1 are true where n is the order of a finite loop. Hence the claims follow.

n	5	6	7
$\lvert\mathcal{L}_n\rvert$	56	9, 408	16, 942, 080
$\{\mathcal{A}_i^n\}_{i=1}^k$	$k = 6$	$k = 109$	$k = 23,746$
$\{\mathfrak{A}_i^n\}_{i=1}^m$	$m = 2$	$m = 22$	$m = 564$

Table 5.1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

Question 5.1.1 *How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{SL}_n|$ or $|\mathcal{NSL}_n|$.*

Theorem 5.1.2 *Let $(G, \cdot)$ be a finite S-groupoid of order n with a finite S-subsemigroup $(H, \cdot)$ of order m. Also, let*

$$\mathcal{ISOT}(G, \cdot),\ \mathcal{SISOT}(G, \cdot) \text{ and } \mathcal{NSISOT}(G, \cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G, \cdot)$. Then,

$$\mathcal{ISOT}(G, \cdot) \text{ is a group and } \mathcal{SISOT}(G, \cdot) \leq \mathcal{ISOT}(G, \cdot).$$

Furthermore:

1. $|\mathcal{ISOT}(G, \cdot)| = (n!)^3$;
2. $|\mathcal{SISOT}(G, \cdot)| = (m!)^3$;
3. $|\mathcal{NSISOT}(G, \cdot)| = (n!)^3 - (m!)^3$.

Proof

1. This has been shown to be true in [Theorem 4.1.1, [29]].

2. An S-isotopism is an isotopism. So, $\mathcal{SISOT}(G,\cdot) \subset \mathcal{ISOT}(G,\cdot)$. Thus, we need to just verify the axioms of a group to show that $\mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 5.1.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G,\cdot) \text{ such that } \alpha = (U,V,W) \ : \ (G,\cdot) \longrightarrow (G,\circ),$$

where $(G,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subgroups $(H,\cdot)$ and $(K,\circ)$ respectively, we can set

$$U' := U|_H, \ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K \ \forall \ A \in \{U,V,W\},$$

so that $\mathcal{SISOT}(H,\cdot) = \{(U',V',W')\}$. This is possible because of the following arguments.

Let

$$X = \Big\{ f' := f|_H \ \Big| \ f : G \longrightarrow G, \ f : H \longrightarrow K \text{ is bijective and } f(H) = K \Big\}.$$

Let

$$SYM(H,K) = \{\text{bijections from } H \text{ unto } K\}.$$

By definition, it is easy to see that $X \subseteq SYM(H,K)$. Now, for all $U \in SYM(H,K)$, define $U \ : \ H^c \longrightarrow K^c$ so that $U \ : \ G \longrightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H,K) \subseteq X$ so that $SYM(H,K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|\mathcal{ISOT}(H,\cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G,\cdot)| = (m!)^3 \text{ since } SYM(H,K) = X.$$

3.

$$\mathcal{NSISOT}(G,\cdot) = \big(\mathcal{SISOT}(G,\cdot)\big)^{c}.$$

So, the identity isotopism

$$(I,I,I) \notin \mathcal{NSISOT}(G,\cdot), \text{ hence } \mathcal{NSISOT}(G,\cdot) \nleq \mathcal{ISOT}(G,\cdot).$$

Furthermore,

$$|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3.$$

Corollary 5.1.2 *Let $(G,\cdot)$ be a finite S-groupoid of order n with an S-subsemigroup $(H,\cdot)$. If $\mathcal{ISOT}(G,\cdot)$ is the group of all isotopisms of $(G,\cdot)$ and S_n is the symmetric group of degree n, then*

$$\mathcal{ISOT}(G,\cdot) \gtrsim S_n \times S_n \times S_n.$$

Proof

As concluded in [Corollary 1, [29]], $\mathcal{ISOT}(G,\cdot) \cong S_n \times S_n \times S_n$. Let $\mathcal{PISOT}(G,\cdot)$ be the set of all principal isotopisms on $(G,\cdot)$. $\mathcal{PISOT}(G,\cdot)$ is an S-subgroup in $\mathcal{ISOT}(G,\cdot)$ while $S_n \times S_n \times \{I\}$ is an S-subgroup in $S_n \times S_n \times S_n$. If

$$\Upsilon \ : \ \mathcal{ISOT}(G,\cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon\big((A,B,I)\big) =< A,B,I > \ \forall \ (A,B,I) \in \mathcal{ISOT}(G,\cdot),$$

then

$$\Upsilon\Big(\mathcal{PISOT}(G,\cdot)\Big) = S_n \times S_n \times \{I\}. \ \therefore \ \mathcal{ISOT}(G,\cdot) \gtrsim S_n \times S_n \times S_n.$$

5.2 Smarandache f, g-Isotopes Of Smarandache Loops

Theorem 5.2.1 *Let $(G, \cdot)$ and $(H, *)$ be S-groupoids. If $(G, \cdot)$ and $(H, *)$ are S-isotopic, then $(H, *)$ is S-isomorphic to some Smarandache principal isotope $(G, \circ)$ of $(G, \cdot)$.*

Proof

Since $(G, \cdot)$ and $(H, *)$ are S-isotopic S-groupoids with S-subsemigroups $(G_1, \cdot)$ and $(H_1, *)$, then there exist bijections $U, V, W : (G, \cdot) \to (H, *)$ such that the triple

$$\alpha = (U, V, W) \ : \ (G, \cdot) \to (H, *) \text{ is an isotopism and } \big(G_1\big)A = H_1 \ \forall \ A \in \{U, V, W\}.$$

To prove the claim of this theorem, it suffices to produce a closed binary operation '$\circ$' on G, bijections $X, Y \ : \ G \to G$, and bijection $Z \ : \ G \to H$ so that

- the triple $\beta = (X, Y, I) \ : \ (G, \cdot) \to (G, \circ)$ is a Smarandache principal isotopism and
- $Z \ : \ (G, \circ) \to (H, *)$ is an S-isomorphism or the triple $\gamma = (Z, Z, Z) \ : \ (G, \circ) \to (H, *)$ is an S-isotopism.

Thus, we need $(G, \circ)$ so that the commutative diagram below is true:

$$(G, \cdot) \xrightarrow[\text{principal isotopism}]{\beta} (G, \circ) \xrightarrow[\text{isomorphism}]{\gamma} (H, *) \xleftarrow[\text{isotopism}]{\alpha} (G, \cdot)$$

because following the proof of transitivity in Theorem 5.1.1, $\alpha = \beta\gamma$ which implies $(U, V, W) = (XZ, YZ, Z)$ and so we can make the choices; $Z = W$, $Y = VW^{-1}$, and $X = UW^{-1}$ and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \ \forall \ x, y \in G.$$

Hence, $(G, \circ)$ is a groupoid principal isotope of $(G, \cdot)$ and $(H, *)$ is an isomorph of $(G, \circ)$. It remains to show that these two relationships are Smarandache.

Note that $\big((H_1)Z^{-1}, \circ\big) = (G_1, \circ)$ is a non-trivial subsemigroup in $(G, \circ)$. Thus, $(G, \circ)$ is an S-groupoid. So $(G, \circ) \succsim (H, *)$. $(G, \cdot)$ and $(G, \circ)$ are Smarandache principal isotopes because

$$(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1 \text{ and } (G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1.$$

Corollary 5.2.1 *Let $(G, \cdot)$ be an S-groupoid with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-groupoid if and only if all the principal isotopes of $(G, \cdot)$ are S-groupoids.*

Proof

By classical result in principal isotopy [[72], III.1.4 Theorem], if $(G, \cdot)$ and $(H, *)$ are isotopic groupoids, then $(H, *)$ is isomorphic to some principal isotope $(G, \circ)$ of $(G, \cdot)$. Assuming $(H, *)$ is an S-groupoid then since $(H, *) \cong (G, \circ)$, $(G, \circ)$ is an S-groupoid. Conversely, let us assume all the principal isotopes of $(G, \cdot)$ are S-groupoids. Since $(H, *) \cong (G, \circ)$, then $(H, *)$ is an S-groupoid.

Theorem 5.2.2 *Let $(G, \cdot)$ be an S-quasigroup. If $(H, *)$ is an S-loop which is S-isotopic to $(G, \cdot)$, then there exist S-elements f and g so that $(H, *)$ is S-isomorphic to a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$.*

Proof

An S-quasigroup and an S-loop are S-groupoids. So by Theorem 5.2.1, $(H, *)$ is S-isomorphic to a Smarandache principal isotope $(G, \circ)$ of $(G, \cdot)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Since $(H, *)$ is a S-loop and $(G, \circ) \succsim (H, *)$ implies that $(G, \circ) \cong (H, *)$, then $(G, \circ)$ is necessarily an S-loop and consequently, $(G, \circ)$ has a two-sided identity element say e and an S-subgroup $(G_2, \circ)$. Let $\alpha = (U, V, I)$ be the Smarandache

principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Then,

$$xU \circ yV = x \cdot y \ \forall \ x, y \in G \iff \ x \circ y = xU^{-1} \cdot yV^{-1} \ \forall \ x, y \in G.$$

So,

$y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1}L_{eU^{-1}} \ \forall \ y \in G$ and $x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1}R_{eV^{-1}} \ \forall \ x \in G.$

Assign $f = eU^{-1}, g = eV^{-1} \in G_2$. This assignments are well defined and hence $V = L_f$ and $U = R_g$. So that $\alpha = (R_g, L_f, I)$ is a Smarandache f, g principal isotopism of $(G, \circ)$ onto $(G, \cdot)$. This completes the proof.

Corollary 5.2.2 *Let $(G, \cdot)$ be an S-quasigroup(S-loop) with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-quasigroup(S-loop) if and only if all the principal isotopes of $(G, \cdot)$ are S-quasigroups(S-loops).*

Proof

This follows immediately from Corollary 5.2.1 since an S-quasigroup and an S-loop are S-groupoids.

Corollary 5.2.3 *If $(G, \cdot)$ and $(H, *)$ are S-loops which are S-isotopic, then there exist S-elements f and g so that $(H, *)$ is S-isomorphic to a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$.*

Proof

An S-loop is an S-quasigroup. So the claim follows from Theorem 5.2.2.

5.3 G-Smarandache Loops

Lemma 5.3.1 *Let $(G,\cdot)$ and $(H,*)$ be S-isotopic S-loops. If $(G,\cdot)$ is a group, then $(G,\cdot)$ and $(H,*)$ are S-isomorphic groups.*

Proof

By Corollary 5.2.3, there exist S-elements f and g in $(G,\cdot)$ so that $(H,*) \succsim (G,\circ)$ such that $(G,\circ)$ is a Smarandache f,g principal isotope of $(G,\cdot)$. Let us set the mapping $\psi := R_{f\cdot g} = R_{fg} : G \to G$. This mapping is bijective. Now, let us consider when $\psi := R_{fg} : (G,\cdot) \to (G,\circ)$. Since $(G,\cdot)$ is associative and $x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x,y \in G$, the following arguments are true.

$$x\psi \circ y\psi = x\psi R_g^{-1} \cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1} \cdot yR_{fg}L_f^{-1} =$$
$$x \cdot fg \cdot g^{-1} \cdot f^{-1} \cdot y \cdot fg = x \cdot y \cdot fg = (x \cdot y)R_{fg} = (x \cdot y)\psi \ \forall \ x,y \in G.$$

So, $(G,\cdot) \cong (G,\circ)$. Thus, $(G,\circ)$ is a group. If $(G_1,\cdot)$ and $(G_1,\circ)$ are the S-subgroups in $(G,\cdot)$ and $(G,\circ)$, then $\big((G_1,\cdot)\big)R_{fg} = (G_1,\circ)$. Hence, $(G,\cdot) \succsim (G,\circ)$.

$$\therefore \ (G,\cdot) \succsim (H,*) \text{ and } (H,*) \text{ is a group.}$$

Corollary 5.3.1 *Every group which is an S-loop is a GS-loop.*

Proof

This follows immediately from Lemma 5.3.1 and the fact that a group is a G-loop.

Corollary 5.3.2 *An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.*

Proof

Let $(G, \cdot)$ be an S-loop with arbitrary S-isotope $(H, *)$. Let us assume that $(G, \cdot) \succsim (H, *)$. From Corollary 5.2.3, for any arbitrary S-isotope $(H, *)$ of $(G, \cdot)$, there exists a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$ such that $(H, *) \succsim (G, \circ)$. So, $(G, \cdot) \succsim (G, \circ)$.

Conversely, let $(G, \cdot) \succsim (G, \circ)$, using the fact in Corollary 5.2.3 again, for any arbitrary S-isotope $(H, *)$ of $(G, \cdot)$, there exists a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$ such that $(G, \circ) \succsim (H, *)$. Therefore, $(G, \cdot) \succsim (H, *)$.

Corollary 5.3.3 *A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache f, g principal isotopes.*

Proof

This follows by the definition of a GS-loop and Corollary 5.3.2.

Chapter 6

SMARANDACHE BRYANT SCHNEIDER GROUP OF SMARANDACHE LOOPS

6.1 Smarandache Special Maps

Definition 6.1.1 *Let $(G, \cdot)$ be a Smarandache loop with S-subgroup $(H, \cdot)$. A mapping $\theta \in SSYM(G, \cdot)$ is a Smarandache special map(S-special map) for G if and only if there exist $f, g \in H$ such that*

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot).$$

Definition 6.1.2 *Let the set*

$$SBS(G, \cdot) = \{\theta \in SSYM(G, \cdot) \ : \ \text{there exist } f, g \in H \ \ni \ (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)\}$$

i.e the set of all S-special maps in a S-loop, then $SBS(G, \cdot)$ is called the Smarandache Bryant-Schneider group(SBS group) of the S-loop $(G, \cdot)$ with S-subgroup H if $SBS(G, \cdot) \leq$

$SYM(G,\cdot)$.

Definition 6.1.3 *Let $(G,\cdot)$ be a Smarandache loop with an S-subgroup H.*

$$\Omega(G,\cdot)=\left\{(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot) \text{ for some } f,g\in H \ : \ h\theta\in H \ \forall \ h\in H\right\}.$$

6.2 The Smarandache Bryant Schneider Group

Theorem 6.2.1 *Let $(G,\cdot)$ be a Smarandache loop. $SBS(G,\cdot)\leq BS(G,\cdot)$.*

Proof

Let $(G,\cdot)$ be an S-loop with S-subgroup H. Comparing the definition of the Bryant-Schneider group of a loop and Definition 6.1.2, it can easily be observed that $SBS(G,\cdot)\subset BS(G,\cdot)$. The case $SBS(G,\cdot)\subseteq BS(G,\cdot)$ is possible when $G=H$ where H is the S-subgroup of G but this will be a contradiction since G is an S-loop.

Identity If I is the identity mapping on G, then $hI=h\in H\ \forall\ h\in H$ and there exists $e\in H$ where e is the identity element in G such that $(IR_e^{-1},IL_e^{-1},I)=(I,I,I)\in AUT(G,\cdot)$. So, $I\in SBS(G,\cdot)$. Thus $SBS(G,\cdot)$ is non-empty.

Closure and Inverse Let $\alpha,\beta\in SBS(G,\cdot)$. Then there exist $f_1,g_1,f_2,g_2\in H$ such that

$$A=(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha),\ B=(\beta R_{g_2}^{-1},\beta L_{f_2}^{-1},\beta)\in AUT(G,\cdot).$$

$$AB^{-1}=(\alpha R_{g_1}^{-1},\alpha L_{f_1}^{-1},\alpha)(R_{g_2}\beta^{-1},L_{f_2}\beta^{-1},\beta^{-1})$$

$$=(\alpha R_{g_1}^{-1}R_{g_2}\beta^{-1},\alpha L_{f_1}^{-1}L_{f_2}\beta^{-1},\alpha\beta^{-1})\in AUT(G,\cdot).$$

Let $\delta=\beta R_{g_1}^{-1}R_{g_2}\beta^{-1}$ and $\gamma=\beta L_{f_1}^{-1}L_{f_2}\beta^{-1}$. Then,

$$(\alpha\beta^{-1}\delta,\alpha\beta^{-1}\gamma,\alpha\beta^{-1})\in AUT(G,\cdot)\Leftrightarrow(x\alpha\beta^{-1}\delta)\cdot(y\alpha\beta^{-1}\gamma)=(x\cdot y)\alpha\beta^{-1}\ \forall\ x,y\in G.$$

Putting $y = e$ and replacing x by $x\beta\alpha^{-1}$, we have $(x\delta)\cdot(e\alpha\beta^{-1}\gamma) = x$ for all $x \in G$. Similarly, putting $x = e$ and replacing y by $y\beta\alpha^{-1}$, we have $(e\alpha\beta^{-1}\delta)\cdot(y\gamma) = y$ for all $y \in G$. Thence, $x\delta R_{(e\alpha\beta^{-1}\gamma)} = x$ and $y\gamma L_{(e\alpha\beta^{-1}\delta)} = y$ which implies that

$$\delta = R^{-1}_{(e\alpha\beta^{-1}\gamma)} \text{ and } \gamma = L^{-1}_{(e\alpha\beta^{-1}\delta)}.$$

Thus, since $g = e\alpha\beta^{-1}\gamma,\ f = e\alpha\beta^{-1}\delta \in H$ then

$$AB^{-1} = (\alpha\beta^{-1}R_g^{-1}, \alpha\beta^{-1}L_f^{-1}, \alpha\beta^{-1}) \in AUT(G,\cdot) \Leftrightarrow \alpha\beta^{-1} \in SBS(G,\cdot).$$

$\therefore SBS(G,\cdot) \leq BS(G,\cdot)$.

Corollary 6.2.1 *Let $(G,\cdot)$ be a Smarandache loop. Then, $SBS(G,\cdot) \leq SSYM(G,\cdot) \leq SYM(G,\cdot)$. Hence, $SBS(G,\cdot)$ is the Smarandache Bryant-Schneider group(SBS group) of the S-loop $(G,\cdot)$.*

Proof

Although the fact that $SBS(G,\cdot) \leq SYM(G,\cdot)$ follows from Theorem 6.2.1 and the fact in [Theorem 1, [77]] that $BS(G,\cdot) \leq SYM(G,\cdot)$. Nevertheless, it can also be traced from the facts that

$$SBS(G,\cdot) \leq SSYM(G,\cdot) \text{ and } SSYM(G,\cdot) \leq SYM(G,\cdot).$$

It is easy to see that

$$SSYM(G,\cdot) \subset SYM(G,\cdot) \text{ and that } SBS(G,\cdot) \subset SSYM(G,\cdot)$$

while the trivial cases

$$SSYM(G,\cdot) \subseteq SYM(G,\cdot) \text{ and } SBS(G,\cdot) \subseteq SSYM(G,\cdot)$$

will contradict the fact that G is an S-loop because these two are possible if the S-subgroup H is G. Reasoning through the axioms of a group, it is easy to show that $SSYM(G,\cdot) \leq SYM(G,\cdot)$. By using the same steps in Theorem 6.2.1, it will be seen that $SBS(G,\cdot) \leq SSYM(G,\cdot)$.

6.3 The SBS Group Of A Smarandache f,g-Principal Isotope

Theorem 6.3.1 *Let $(G,\cdot)$ be a S-loop with a Smarandache f,g-principal isotope $(G,\circ)$. Then, $(G,\circ)$ is an S-loop.*

Proof

Let $(G,\cdot)$ be an S-loop, then there exist an S-subgroup $(H,\cdot)$ of G. If $(G,\circ)$ is a Smarandache f,g-principal isotope of $(G,\cdot)$, then

$$x \cdot y = xR_g \circ yL_f \ \forall \ x,y \in G \text{ which implies } x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x,y \in G$$

where $f,g \in H$. So

$$h_1 \circ h_2 = h_1R_g^{-1} \cdot h_2L_f^{-1} \ \forall \ h_1,h_2 \in H \text{ for some } f,g \in H.$$

Let us now consider the set H under the operation "$\circ$". That is the pair $(H,\circ)$.

Groupoid Since $f,g \in H$, then by the definition

$$h_1 \circ h_2 = h_1R_g^{-1} \cdot h_2L_f^{-1}, \ h_1 \circ h_2 \in H \ \forall \ h_1,h_2 \in H$$

since $(H,\cdot)$ is a groupoid. Thus, $(H,\circ)$ is a groupoid.

Quasigroup With the definition

$$h_1 \circ h_2 = h_1 R_g^{-1} \cdot h_2 L_f^{-1} \ \forall \ h_1, h_2 \in H,$$

it is clear that $(H, \circ)$ is a quasigroup since $(H, \cdot)$ is a quasigroup.

Loop It can easily be seen that $f \cdot g$ is an identity element in $(H, \circ)$. So, $(H, \circ)$ is a loop.

Group Since $(H, \cdot)$ is associative, it is easy to show that $(H, \circ)$ is associative.

Hence, $(H, \circ)$ is an S-subgroup in $(G, \circ)$ since the latter is a loop(a quasigroup with identity element $f \cdot g$). Therefore, $(G, \circ)$ is an S-loop.

Theorem 6.3.2 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup $(H, \cdot)$. A mapping $\theta \in SYM(G, \cdot)$ is a S-special map if and only if θ is an S-isomorphism of $(G, \cdot)$ onto some Smarandache f, g-principal isotopes $(G, \circ)$ where $f, g \in H$.*

Proof

By Definition 6.1.1, a mapping $\theta \in SSYM(G)$ is a S-special map implies there exist $f, g \in H$ such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$. It can be observed that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I) \in AUT(G, \cdot).$$

But since

$$(R_g^{-1}, L_f^{-1}, I) \ : \ (G, \circ) \longrightarrow (G, \cdot) \text{ then for } (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$$

we must have

$$(\theta, \theta, \theta) \ : \ (G, \cdot) \longrightarrow (G, \circ) \text{ which means } (G, \cdot) \cong^{\theta} (G, \circ),$$

hence

$$(G,\cdot) \succsim^{\theta} (G,\circ) \text{ because } (H,\cdot)\theta = (H,\circ).\ (R_g, L_f, I)\ :\ (G,\cdot) \longrightarrow (G,\circ)$$

is an f,g-principal isotopism so $(G,\circ)$ is a Smarandache f,g-principal isotope of $(G,\cdot)$ by Theorem 6.3.1.

Conversely, if θ is an S-isomorphism of $(G,\cdot)$ onto some Smarandache f,g-principal isotopes $(G,\circ)$ where $f,g \in H$ such that $(H,\cdot)$ is a S-subgroup of $(G,\cdot)$ means

$$(\theta,\theta,\theta)\ :\ (G,\cdot) \longrightarrow (G,\circ),\ (R_g, L_f, I)\ :\ (G,\cdot) \longrightarrow (G,\circ)$$

which implies

$$(R_g^{-1}, L_f^{-1}, I)\ :\ (G,\circ) \longrightarrow (G,\cdot) \text{ and } (H,\cdot)\theta = (H,\circ).$$

Thus, $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)$. Therefore, θ is a S-special map because $f,g \in H$.

Corollary 6.3.1 *Let $(G,\cdot)$ be a Smarandache loop with a an S-subgroup $(H,\cdot)$. A mapping $\theta \in SBS(G,\cdot)$ if and only if θ is an S-isomorphism of $(G,\cdot)$ onto some Smarandache f,g-principal isotopes $(G,\circ)$ such that $f,g \in H$ where $(H,\cdot)$ is an S-subgroup of $(G,\cdot)$.*

Proof

This follows from Definition 6.1.2 and Theorem 6.3.2.

Theorem 6.3.3 *Let $(G,\cdot)$ and $(G,\circ)$ be S-loops. $(G,\circ)$ is a Smarandache f,g-principal isotope of $(G,\cdot)$ if and only if $(G,\cdot)$ is a Smarandache g,f-principal isotope of $(G,\circ)$.*

Proof

Let $(G,\cdot)$ and $(G,\circ)$ be S-loops such that if $(H,\cdot)$ is an S-subgroup in $(G,\cdot)$, then $(H,\circ)$ is an S-subgroup of $(G,\circ)$. The left and right translation maps relative to an element x in $(G,\circ)$ shall be denoted by $\mathcal{L}_x$ and $\mathcal{R}_x$ respectively.

If $(G, \circ)$ is a Smarandache f, g-principal isotope of $(G, \cdot)$ then,

$$x \cdot y = xR_g \circ yL_f \ \forall \ x, y \in G \text{ for some } f, g \in H.$$

Thus,

$$xR_y = xR_g\mathcal{R}_{yL_f} \text{ and } yL_x = yL_f\mathcal{L}_{xR_g}, \ x, y \in G$$

and we have

$$R_y = R_g\mathcal{R}_{yL_f} \text{ and } L_x = L_f\mathcal{L}_{xR_g}, \ x, y \in G.$$

So,

$$\mathcal{R}_y = R_g^{-1}R_{yL_f^{-1}} \text{ and } \mathcal{L}_x = L_f^{-1}L_{xR_g^{-1}}, \ x, y \in G.$$

Putting $y = f$ and $x = g$ respectively, we now get

$$\mathcal{R}_f = R_g^{-1}R_{fL_f^{-1}} = R_g^{-1} \text{ and } \mathcal{L}_g = L_f^{-1}L_{gR_g^{-1}} = L_f^{-1}.$$

That is,

$$\mathcal{R}_f = R_g^{-1} \text{ and } \mathcal{L}_g = L_f^{-1} \text{ for some } f, g \in H.$$

Recall that

$$x \cdot y = xR_g \circ yL_f \ \forall \ x, y \in G \ \Leftrightarrow \ x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x, y \in G.$$

So using the last two translation equations,

$$x \circ y = x\mathcal{R}_f \cdot y\mathcal{L}_g \ \forall \ x, y \in G \ \Leftrightarrow \text{ the triple } (\mathcal{R}_f, \mathcal{L}_g, I) \ : \ (G, \circ) \longrightarrow (G, \cdot)$$

is a Smarandache g, f-principal isotopism. Therefore, $(G, \cdot)$ is a Smarandache g, f-principal isotope of $(G, \circ)$.

The proof of the converse is achieved by doing the reverse of the procedure described above.

Theorem 6.3.4 *If $(G,\cdot)$ is an S-loop with a Smarandache f,g-principal isotope $(G,\circ)$, then $SBS(G,\cdot) = SBS(G,\circ)$.*

Proof

Let $(G,\circ)$ be the Smarandache f,g-principal isotope of the S-loop $(G,\cdot)$ with S-subgroup $(H,\cdot)$. By Theorem 6.3.1, $(G,\circ)$ is an S-loop with S-subgroup $(H,\circ)$. The left and right translation maps relative to an element x in $(G,\circ)$ shall be denoted by $\mathcal{L}_x$ and $\mathcal{R}_x$ respectively.

Let $\alpha \in SBS(G,\cdot)$, then there exist $f_1, g_1 \in H$ so that

$$(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \in AUT(G,\cdot).$$

Recall that the triple

$$(R_{g_1}, L_{f_1}, I) \ : \ (G,\cdot) \longrightarrow (G,\circ)$$

is a Smarandache f,g-principal isotopism, so

$$x \cdot y = xR_g \circ yL_f \ \forall \ x,y \in G$$

and this implies

$$R_x = R_g \mathcal{R}_{xL_f} \text{ and } L_x = L_f \mathcal{L}_{xR_g} \ \forall \ x \in G \text{ which also implies that}$$

$$\mathcal{R}_{xL_f} = R_g^{-1} R_x \text{ and } \mathcal{L}_{xR_g} = L_f^{-1} L_x \ \forall \ x \in G \text{ which finally gives}$$

$$\mathcal{R}_x = R_g^{-1} R_{xL_f^{-1}} \text{ and } \mathcal{L}_x = L_f^{-1} L_{xR_g^{-1}} \ \forall \ x \in G.$$

Set

$$f_2 = f\alpha R_{g_1}^{-1} R_g \text{ and } g_2 = g\alpha L_{f_1}^{-1} L_f.$$

Then

$$\mathcal{R}_{g_2} = R_g^{-1} R_{g\alpha L_{f_1}^{-1} L_f L_f^{-1}} = R_g^{-1} R_{g\alpha L_{f_1}^{-1}} \tag{6.1}$$

$$\text{and } \mathcal{L}_{f_2} = L_f^{-1} L_{f\alpha R_{g_1}^{-1} R_g R_g^{-1}} = L_f^{-1} L_{f\alpha R_{g_1}^{-1}} \ \forall\ x \in G. \tag{6.2}$$

Since, $(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \in AUT(G, \cdot)$, then

$$(x\alpha R_{g_1}^{-1}) \cdot (y\alpha L_{f_1}^{-1}) = (x \cdot y)\alpha \ \forall\ x, y \in G. \tag{6.3}$$

Putting $y = g$ and $x = f$ separately in the last equation,

$$x\alpha R_{g_1}^{-1} R_{(g\alpha L_{f_1}^{-1})} = xR_g\alpha \text{ and } y\alpha L_{f_1}^{-1} L_{(f\alpha R_{g_1}^{-1})} = yL_f\alpha \ \forall\ x, y \in G.$$

Thus by applying (6.1) and (6.2), we now have

$$\alpha R_{g_1}^{-1} = R_g \alpha R_{(g\alpha L_{f_1}^{-1})}^{-1} = R_g \alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \text{ and } \alpha L_{f_1}^{-1} = L_f \alpha L_{(f\alpha R_{g_1}^{-1})}^{-1} = L_f \alpha \mathcal{L}_{f_2}^{-1} L_f^{-1}. \tag{6.4}$$

We shall now compute $(x \circ y)\alpha$ by (6.3) and (6.4) and then see the outcome.

$$(x \circ y)\alpha = (xR_g^{-1} \cdot yL_f^{-1})\alpha = xR_g^{-1}\alpha R_{g_1}^{-1} \cdot yL_f^{-1}\alpha L_{f_1}^{-1} =$$
$$xR_g^{-1} R_g \alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \cdot yL_f^{-1} L_f \alpha \mathcal{L}_{f_2}^{-1} L_f^{-1} =$$
$$x\alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \cdot y\alpha \mathcal{L}_{f_2}^{-1} L_f^{-1} = x\alpha \mathcal{R}_{g_2}^{-1} \circ y\alpha \mathcal{L}_{f_2}^{-1} \ \forall\ x, y \in G.$$

Thus,

$$(x \circ y)\alpha = x\alpha \mathcal{R}_{g_2}^{-1} \circ y\alpha \mathcal{L}_{f_2}^{-1} \ \forall\ x, y \in G \Leftrightarrow (\alpha \mathcal{R}_{g_2}^{-1}, \alpha \mathcal{L}_{f_2}^{-1}, \alpha) \in AUT(G, \circ) \Leftrightarrow \alpha \in SBS(G, \circ).$$

Whence, $SBS(G,\cdot) \subseteq SBS(G,\circ)$.

Since $(G,\circ)$ is the Smarandache f,g-principal isotope of the S-loop $(G,\cdot)$, then by Theorem 6.3.3, $(G,\cdot)$ is the Smarandache g,f-principal isotope of $(G,\circ)$. So following the steps above, it can similarly be shown that $SBS(G,\circ) \subseteq SBS(G,\cdot)$. Therefore, the conclusion that $SBS(G,\cdot) = SBS(G,\circ)$ follows.

6.4 Cardinality Formulas

Theorem 6.4.1 *Let $(G,\cdot)$ be a finite Smarandache loop with n distinct S-subgroups. If the SBS group of $(G,\cdot)$ relative to an S-subgroup $(H_i,\cdot)$ is denoted by $SBS_i(G,\cdot)$, then*

$$|BS(G,\cdot)| = \frac{1}{n}\sum_{i=1}^{n} |SBS_i(G,\cdot)|\ [BS(G,\cdot) : SBS_i(G,\cdot)].$$

Proof

Let the n distinct S-subgroups of G be denoted by $H_i,\ i = 1,2,\cdots n$. Note here that $H_i \neq H_j\ \forall\ i,j = 1,2,\cdots n$. By Theorem 6.2.1, $SBS_i(G,\cdot) \leq BS(G,\cdot)\ \forall\ i = 1,2,\cdots n$. Hence, by the Lagrange's theorem of classical group theory,

$$|BS(G,\cdot)| = |SBS_i(G,\cdot)|\ [BS(G,\cdot) : SBS_i(G,\cdot)]\ \forall\ i = 1,2,\cdots n.$$

Thus, adding the equation above for all $i = 1,2,\cdots n$, we get

$$n|BS(G,\cdot)| = \sum_{i=1}^{n} |SBS_i(G,\cdot)|\ [BS(G,\cdot) : SBS_i(G,\cdot)]\ \forall\ i = 1,2,\cdots n,\ \text{thence,}$$

$$|BS(G,\cdot)| = \frac{1}{n}\sum_{i=1}^{n} |SBS_i(G,\cdot)|\ [BS(G,\cdot) : SBS_i(G,\cdot)].$$

Theorem 6.4.2 *Let $(G,\cdot)$ be a Smarandache loop. Then, $\Omega(G,\cdot) \leq AUT(G,\cdot)$.*

Proof

Let $(G, \cdot)$ be an S-loop with S-subgroup H. By Definition 6.1.3, it can easily be observed that $\Omega(G, \cdot) \subseteq AUT(G, \cdot)$.

Identity If I is the identity mapping on G, then $hI = h \in H \ \forall \ h \in H$ and there exists $e \in H$ where e is the identity element in G such that

$$(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(G, \cdot).$$

So, $(I, I, I) \in \Omega(G, \cdot)$. Thus $\Omega(G, \cdot)$ is non-empty.

Closure and Inverse Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), \ B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$$AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1})$$

$$= (\alpha R_{g_1}^{-1} R_{g_2}\beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2}\beta^{-1}, \alpha\beta^{-1}) \in AUT(G, \cdot).$$

Using the same techniques for the proof of closure and inverse in Theorem 6.2.1 here and by letting

$$\delta = \beta R_{g_1}^{-1} R_{g_2}\beta^{-1} \text{ and } \gamma = \beta L_{f_1}^{-1} L_{f_2}\beta^{-1},$$

it can be shown that,

$$AB^{-1} = (\alpha\beta^{-1}R_g^{-1}, \alpha\beta^{-1}L_f^{-1}, \alpha\beta^{-1}) \in AUT(G, \cdot) \text{ where } g = e\alpha\beta^{-1}\gamma, \ f = e\alpha\beta^{-1}\delta \in H$$

$$\text{such that } \alpha\beta^{-1} \in SSYM(G, \cdot) \Leftrightarrow AB^{-1} \in \Omega(G, \cdot).$$

$\therefore \Omega(G, \cdot) \leq AUT(G, \cdot)$.

Theorem 6.4.3 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup H such that $f, g \in H$ and $\alpha \in SBS(G, \cdot)$. If the mapping*

$$\Phi \ : \ \Omega(G, \cdot) \longrightarrow SBS(G, \cdot) \ \ \textit{is defined as} \ \ \Phi \ : \ (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha,$$

then Φ is an homomorphism.

Proof

Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), \ B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$$\Phi(AB) = \Phi[(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta)] =$$

$$\Phi(\alpha R_{g_1}^{-1} \beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1} \beta L_{f_2}^{-1}, \alpha\beta).$$

It will be good if this can be written as;

$$\Phi(AB) = \Phi(\alpha\beta\delta, \alpha\beta\gamma, \alpha\beta) \text{ such that } h\alpha\beta \in H \ \forall \ h \in H$$

and $\delta = R_g^{-1}$, $\gamma = L_f^{-1}$ for some $g, f \in H$.

This is done as follows: If

$$(\alpha R_{g_1}^{-1} \beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1} \beta L_{f_2}^{-1}, \alpha\beta) = (\alpha\beta\delta, \alpha\beta\gamma, \alpha\beta) \in AUT(G, \cdot) \text{ then,}$$

$$x\alpha\beta\delta \cdot y\alpha\beta\gamma = (x \cdot y)\alpha\beta \ \forall \ x, y \in G.$$

Put $y = e$ and replace x by $x\beta^{-1}\alpha^{-1}$ then

$$x\delta \cdot e\alpha\beta\gamma = x \Leftrightarrow \delta = R^{-1}_{e\alpha\beta\gamma}.$$

Similarly, put $x = e$ and replace y by $y\beta^{-1}\alpha^{-1}$. Then,

$$e\alpha\beta\delta \cdot y\gamma = y \Leftrightarrow \gamma = L^{-1}_{e\alpha\beta\delta}.$$

So,

$$\Phi(AB) = (\alpha\beta R^{-1}_{e\alpha\beta\gamma}, \alpha\beta L^{-1}_{e\alpha\beta\delta}, \alpha\beta) = \alpha\beta = \Phi(\alpha R^{-1}_{g_1}, \alpha L^{-1}_{f_1}, \alpha)\Phi(\beta R^{-1}_{g_2}, \beta L^{-1}_{f_2}, \beta) = \Phi(A)\Phi(B).$$

$\therefore \Phi$ is an homomorphism.

Theorem 6.4.4 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup H such that $f, g \in H$ and $\alpha \in SSYM(G, \cdot)$. If the mapping*

$$\Phi \ : \ \Omega(G, \cdot) \longrightarrow SBS(G, \cdot) \ \textit{ is defined as } \ \Phi \ : \ (\alpha R^{-1}_g, \alpha L^{-1}_f, \alpha) \mapsto \alpha$$

then,

$$A = (\alpha R^{-1}_g, \alpha L^{-1}_f, \alpha) \in \ker \Phi \textit{ if and only if } \alpha$$

is the identity map on G, $g \cdot f$ is the identity element of $(G, \cdot)$ and $g \in N_\mu(G, \cdot)$ the middle nucleus of $(G, \cdot)$.

Proof

Necessity $\ker \Phi = \{A \in \Omega(G, \cdot) : \Phi(A) = I\}$. So, if

$$A = (\alpha R^{-1}_{g_1}, \alpha L^{-1}_{f_1}, \alpha) \in \ker \Phi, \text{ then } \Phi(A) = \alpha = I.$$

Thus, $A = (R_{g_1}^{-1}, L_{f_1}^{-1}, I) \in AUT(G, \cdot) \Leftrightarrow$

$$x \cdot y = xR_g^{-1} \cdot yL_f^{-1} \; \forall \; x, y \in G. \tag{6.5}$$

Replace x by xR_g and y by yL_f in (6.5) to get

$$x \cdot y = xg \cdot fy \; \forall \; x, y \in G. \tag{6.6}$$

Putting $x = y = e$ in (6.6), we get $g \cdot f = e$. Replace y by yL_f^{-1} in (6.6) to get

$$x \cdot yL_f^{-1} = xg \cdot y \; \forall \; x, y \in G. \tag{6.7}$$

Put $x = e$ in (6.7), then we have $yL_f^{-1} = g \cdot y \; \forall \; y \in G$ and so (6.7) now becomes

$$x \cdot (gy) = xg \cdot y \; \forall \; x, y \in G \Leftrightarrow g \in N_\mu(G, \cdot).$$

Sufficiency Let α be the identity map on G, $g \cdot f$ the identity element of $(G, \cdot)$ and $g \in N_\mu(G, \cdot)$. Thus, $fg \cdot f = f \cdot gf = fe = f$. Thus, $f \cdot g = e$. Then also, $y = fg \cdot y = f \cdot gy \; \forall \; y \in G$ which results into $yL_f^{-1} = gy \; \forall \; y \in G$. Thus, it can be seen that

$$x\alpha R_g^{-1} \cdot y\alpha L_f^{-1} = xR_g^{-1} \cdot yL_f^{-1} = xR_g^{-1}\alpha \cdot yL_f^{-1}\alpha =$$
$$xR_g^{-1} \cdot yL_f^{-1} = xR_g^{-1} \cdot gy = (xR_g^{-1} \cdot g)y = xR_g^{-1}R_g \cdot y = x \cdot y = (x \cdot y)\alpha \; \forall \; x, y \in G.$$

Thus,

$$\Phi(A) = \Phi(\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) = \Phi(R_g^{-1}, L_f^{-1}, I) = I \Rightarrow A \in \ker \Phi.$$

Theorem 6.4.5 *Let $(G, \cdot)$ be a Smarandache loop with an S-subgroup H such that $f, g \in H$*

and $\alpha \in SSYM(G,\cdot)$. *If the mapping*

$$\Phi \ : \ \Omega(G,\cdot) \longrightarrow SBS(G,\cdot) \ \ \textit{is defined as} \ \ \Phi \ : \ (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha$$

then,

$$|N_\mu(G,\cdot)| = |\ker \Phi| \ \textit{and} \ |\Omega(G,\cdot)| = |SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Proof

Let the identity map on G be I. Using Theorem 6.4.4, if

$$g\theta = (R_g^{-1}, L_{g^{-1}}^{-1}, I) \ \forall \ g \in N_\mu(G,\cdot) \text{ then, } \ \theta \ : \ N_\mu(G,\cdot) \longrightarrow \ker \Phi.$$

θ is easily seen to be a bijection, hence $|N_\mu(G,\cdot)| = |\ker \Phi|$.

Since Φ is an homomorphism by Theorem 6.4.3, then by the first isomorphism theorem in classical group theory,

$$\Omega(G,\cdot)/\ker \Phi \cong \mathrm{Im}\Phi.$$

Φ is clearly onto, so $\mathrm{Im}\Phi = SBS(G,\cdot)$, so that $\Omega(G,\cdot)/\ker \Phi \cong SBS(G,\cdot)$. Thus, $|\Omega(G,\cdot)/\ker \Phi| = |SBS(G,\cdot)|$. By Lagrange's theorem,

$$|\Omega(G,\cdot)| = |\ker \Phi||\Omega(G,\cdot)/\ker \Phi|,$$

so,

$$|\Omega(G,\cdot)| = |\ker \Phi||SBS(G,\cdot)|, \ \ \therefore |\Omega(G,\cdot)| = |N_\mu(G,\cdot)||SBS(G,\cdot)|.$$

Theorem 6.4.6 *Let* $(G,\cdot)$ *be a Smarandache loop with an S-subgroup* H. *If*

$$\Theta(G,\cdot) = \Big\{ (f,g) \in H \times H \ : \ (G,\circ) \gtrsim (G,\cdot)$$

for $(G,\circ)$ *the Smarandache principal* f,g*-isotope of* $(G,\cdot)\Big\}$ *then,*

$$|\Omega(G,\cdot)| = |\Theta(G,\cdot)||SAUM(G,\cdot)|.$$

Proof

Let $A, B \in \Omega(G,\cdot)$. Then there exist $\alpha, \beta \in SSYM(G,\cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha),\ B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G,\cdot).$$

Define a relation $\sim$ on $\Omega(G,\cdot)$ such that

$$A \sim B \Longleftrightarrow f_1 = f_2 \text{ and } g_1 = g_2.$$

It is very easy to show that $\sim$ is an equivalence relation on $\Omega(G,\cdot)$. It can easily be seen that the equivalence class $[A]$ of $A \in \Omega(G,\cdot)$ is the inverse image of the mapping

$$\Psi\ :\ \Omega(G,\cdot) \longrightarrow \Theta(G,\cdot) \text{ defined as } \Psi\ :\ (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \mapsto (f,g).$$

If $A, B \in \Omega(G,\cdot)$ then $\Psi(A) = \Psi(B)$ if and only if $(f_1, g_1) = (f_2, g_2)$ so, $f_1 = f_2$ and $g_1 = g_2$. Thus, since $\Omega(G,\cdot) \leq AUT(G,\cdot)$ by Theorem 6.4.2, then

$$AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta)^{-1} = (\alpha R_{g_1}^{-1} R_{g_2} \beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2} \beta^{-1}, \alpha\beta^{-1}) =$$
$$(\alpha\beta^{-1}, \alpha\beta^{-1}, \alpha\beta^{-1}) \in AUT(G,\cdot) \Leftrightarrow \alpha\beta^{-1} \in SAUM(G,\cdot).$$

So,

$$A \sim B \Longleftrightarrow \alpha\beta^{-1} \in SAUM(G,\cdot) \text{ and } (f_1, g_1) = (f_2, g_2).$$

$\therefore |[A]| = |SAUM(G,\cdot)|$. But each

$$A = (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in \Omega(G,\cdot)$$

is determined by some $f, g \in H$. So since the set $\left\{[A] \ : \ A \in \Omega(G,\cdot)\right\}$ of all equivalence classes partitions $\Omega(G,\cdot)$ by the fundamental theorem of equivalence relation,

$$|\Omega(G,\cdot)| = \sum_{f,g\in H} |[A]| = \sum_{f,g\in H} |SAUM(G,\cdot)| = |\Theta(G,\cdot)||SAUM(G,\cdot)|.$$

$\therefore |\Omega(G,\cdot)| = |\Theta(G,\cdot)||SAUM(G,\cdot)|$.

Theorem 6.4.7 *Let $(G,\cdot)$ be a finite Smarandache loop with a finite S-subgroup H. $(G,\cdot)$ is S-isomorphic to all its S-loop S-isotopes if and only if*

$$|(H,\cdot)|^2|SAUM(G,\cdot)| = |SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Proof

As shown in [Corollary 5.2, [44]], an S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f, g principal isotopes. This will happen if and only if $H \times H = \Theta(G,\cdot)$ where $\Theta(G,\cdot)$ is as defined in Theorem 6.4.6.

Since $\Theta(G,\cdot) \subseteq H \times H$ then it is easy to see that for a finite Smarandache loop with a finite S-subgroup H, $H \times H = \Theta(G,\cdot)$ if and only if $|H|^2 = |\Theta(G,\cdot)|$. So the proof is complete by Theorem 6.4.5 and Theorem 6.4.6.

Corollary 6.4.1 *Let $(G,\cdot)$ be a finite Smarandache loop with a finite S-subgroup H. $(G,\cdot)$ is a GS-loop if and only if*

$$|(H,\cdot)|^2|SAUM(G,\cdot)| = |SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Proof

This follows by the definition of a GS-loop and Theorem 6.4.7.

Lemma 6.4.1 *Let $(G,\cdot)$ be a finite GS-loop with a finite S-subgroup H and a middle nucleus $N_\mu(G,\cdot)$.*

$$|(H,\cdot)| = |N_\mu(G,\cdot)| \Longleftrightarrow |(H,\cdot)| = \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}.$$

Proof

From Corollary 6.4.1,

$$|(H,\cdot)|^2|SAUM(G,\cdot)| = |SBS(G,\cdot)||N_\mu(G,\cdot)|.$$

Necessity If $|(H,\cdot)| = |N_\mu(G,\cdot)|$, then

$$|(H,\cdot)||SAUM(G,\cdot)| = |SBS(G,\cdot)| \Longrightarrow |(H,\cdot)| = \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}.$$

Sufficiency If $|(H,\cdot)| = \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}$ then, $|(H,\cdot)||SAUM(G,\cdot)| = |SBS(G,\cdot)|$. Hence, multiplying both sides by $|(H,\cdot)|$,

$$|(H,\cdot)|^2|SAUM(G,\cdot)| = |SBS(G,\cdot)||(H,\cdot)|.$$

So that

$$|SBS(G,\cdot)||N_\mu(G,\cdot)| = |SBS(G,\cdot)||(H,\cdot)| \Longrightarrow |(H,\cdot)| = |N_\mu(G,\cdot)|.$$

Corollary 6.4.2 *Let $(G,\cdot)$ be a finite GS-loop with a finite S-subgroup H. If $|N_\mu(G,\cdot)| \gneqq 1$,*

then,

$$|(H,\cdot)| = \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}.\ \textit{Hence,}\ |(G,\cdot)| = \frac{n|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}\ \textit{for some}\ n \gneqq 1.$$

Proof

By hypothesis, $\{e\} \neq H \neq G$. In a loop, $N_\mu(G,\cdot)$ is a subgroup, hence if $|N_\mu(G,\cdot)| \gneqq 1$, then, we can take

$$(H,\cdot) = N_\mu(G,\cdot) \text{ so that } |(H,\cdot)| = |N_\mu(G,\cdot)|.$$

Thus by Lemma 6.4.1,

$$|(H,\cdot)| = \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}.$$

As shown in [Section 1.3, [37]], a loop L obeys the Lagrange's theorem relative to a subloop H if and only if $H(hx) = Hx$ for all $x \in L$ and for all $h \in H$. This condition is obeyed by $N_\mu(G,\cdot)$, hence

$$|(H,\cdot)|\Big||(G,\cdot)| \Longrightarrow \frac{|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}\Big||(G,\cdot)| \Longrightarrow$$

there exists $n \in \mathbb{N}$ such that

$$|(G,\cdot)| = \frac{n|SBS(G,\cdot)|}{|SAUM(G,\cdot)|}.$$

But if $n = 1$, then $|(G,\cdot)| = |(H,\cdot)| \Longrightarrow (G,\cdot) = (H,\cdot)$ hence $(G,\cdot)$ is a group which is a contradiction to the fact that $(G,\cdot)$ is an S-loop.

$$\therefore\ |(G,\cdot)| = \frac{n|SBS(G,\cdot)|}{|SAUM(G,\cdot)|} \text{ for some natural numbers } n \gneqq 1.$$

Bibliography

[1] J. O. Adeniran (2005), *On holomorphic theory of a class of left Bol loops*, Al.I.Cuza 51, 1, 23-28.

[2] J. O. Adeniran (2002), *The study of properties of certain class of loops via their Bryant-Schneider group*, Ph.D thesis, University of Agriculture, Abeokuta.

[3] J. O. Adeniran (2003), *Some properties of the Bryant-Schneider groups of certain Bol loops*, Proc. Jangjeon Math. Soc. 6, 1, 71-80.

[4] J. O. Adeniran (2002), *More on the Bryant-Schneider group of a conjugacy closed loop*, Proc. Jangjeon Math. Soc. 5, 1, 35-46.

[5] J. O. Adeniran (1999), *On the Bryant-Schneider group of a conjugacy closed loop*, Scientific Annals of Al.I.Cuza. Univ. 45, 2, 241-246.

[6] J. O. Adeniran (2004), *On some maps of conjugacy closed loops*, Scientific Annals of Al.I.Cuza. Univ. 50, 2, 267–272.

[7] R. Artzy (1955), *On loops with special property*, Proc. Amer. Math. Soc. 6, 448–453.

[8] R. Artzy (1959), *On Automorphic-Inverse Properties in Loops*, Proc. Amer. Math. Soc. 10,4, 588–591.

[9] R. Artzy (1978), *Inverse-Cycles in Weak-Inverse Loops*, Proc. Amer. Math. Soc. 68, 2, 132–134.

[10] R. Artzy (1963), *Isotopy and Parastrophy of Quasigroups*, Proc. Amer. Math. Soc. 14, 3, 429–431.

[11] R. Artzy (1959), *Crossed inverse and related loops*, Trans. Amer. Math. Soc. 91, 3, 480–492.

[12] A. S. Basarab (1967), *A class of WIP-loops*, Mat. Issled. 2(2), 3-24.

[13] A. S. Basarab(1970), *Isotopy of WIP loops*, Mat. Issled. 5, 2(16), 3-12.

[14] A. S. Basarab (1973), *The Osborn loop*, Studies in the theory of quasigroups and loops, 193. Shtiintsa, Kishinev, 12–18.

[15] A. S. Basarab and A. I. Belioglo (1979), *Universal automorphic inverse G-loops*, Quasigroups and loops, Mat. Issled. 71, 3–7.

[16] A. S. Basarab and A. I. Belioglo (1979), *UAI Osborn loops*, Quasigroups and loops, Mat. Issled. 51, 8–16.

[17] V. D. Belousov (1965), *Systems of quasigroups with generalised identities*, Usp. Mat. Nauk. 20, 1(121), 75–146.

[18] V. D. Belousov (1969), *Crossed inverse quasigroups(CI-quasigroups)*, Izv. Vyss. Ucebn; Zaved. Matematika 82, 21–27.

[19] R. H. Bruck (1944), *Contributions to the theory of loops*, Trans. Amer. Soc. 55, 245–354.

[20] R. H. Bruck (1944), *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. 55, 19–52.

[21] R. H. Bruck (1966), *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 185pp.

[22] R. H. Bruck and L. J. Paige (1956), *Loops whose inner mappings are automorphisms*, The annuals of Mathematics, 63, 2, 308–323.

[23] B. F. Bryant and H. Schneider (1966), *Principal loop-isotopes of quasigroups*, Canad. J. Math. 18, 120–125.

[24] R. E. Cawagas (2000), *Generation of NAFIL loops of small order*, Quasigroups and Related Systems, 7, 1–5.

[25] O. Chein, H. O. Pflugfelder and J. D. H. Smith (1990), *Quasigroups and Loops : Theory and Applications*, Heldermann Verlag, 568pp.

[26] V. O. Chiboka and A. R. T. Solarin (1991), *Holomorphs of conjugacy closed loops*, Scientific Annals of Al.I.Cuza. Univ. 37, 3, 277–284.

[27] V. O. Chiboka and A. R. T. Solarin (1993), *Autotopism characterization of G-loops*, Scientific Annals of Al.I.Cuza. Univ. 39, 1, 19–26.

[28] V. O. Chiboka (1996), *The Bryant-Schneider group of an extra loop*, Collection of Scientific papers of the Faculty of Science, Kragujevac, 18, 9–20.

[29] J. Dene and A. D. Keedwell (1974), *Latin squares and their applications*, the English University press Lts, 549pp.

[30] J. Duplak (2000), *A parastrophic equivalence in quasigroups*, Quasigroups and Related Systems 7, 7–14.

[31] E. Falconer (1969), *Quasigroup identities invariant under isotopy*, Ph.D thesis, Emory University.

[32] E. Falconer (1970), *Isotopy invariants in quasigroups*, Trans. Amer. Math. Soc. 151, 2, 511-526.

[33] F. Fenyves (1968), *Extra loops I*, Publ. Math. Debrecen, 15, 235–238.

[34] F. Fenyves (1969), *Extra loops II*, Publ. Math. Debrecen, 16, 187–192.

[35] R. A. Fisher and F. Yates (1934), *The* 6×6 *Latin squares*, Proc. Camb. Philos. Soc. 30, 429–507.

[36] T. Foguel (2000), *Groups, transversals, and loops*, Comment. Math. Carolinae 41,2, 261–269.

[37] E. G. Goodaire, E. Jespers and C. P. Milies (1996), *Alternative Loop Rings*, NHMS(184), Elsevier, 387pp.

[38] E. G. Goodaire and D. A. Robinson (1982), *A class of loops which are isomorphic to all loop isotopes*, Can. J. Math. 34, 662–672.

[39] E. G. Goodaire and D. A. Robinson (1990), *Some special conjugacy closed loops*, Canad. Math. Bull. 33, 73–78.

[40] E. D. Huthnance Jr.(1968), *A theory of generalised Moufang loops*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin.

[41] T. G. Jaíyéọlá (2006), *An holomorphic study of the Smarandache concept in loops*, Scientia Magna Journal, 2, 1, 1–8.

[42] T. G. Jaíyéọlá (2006), *Parastrophic invariance of Smarandache quasigroups*, Scientia Magna Journal, 2, 3, 48–53.

[43] T. G. Jaíyéọlá (2006), *On the universality of some Smarandache loops of Bol-Moufang type*, Scientia Magna Journal, 2, 4, 45–48.

[44] T. G. Jaíyéọlá (2008), *Smarandache isotopy theory of Smarandache: quasigroups and loops*, Proceedings of the 4th International Conference on Number Theory and Smarandache Problems, Scientia Magna Journal, Vol. 4, No. 1, 168-177.

[45] T. G. Jaíyéọlá (2005), *An isotopic study of properties of central loops*, M.Sc. Dissertation, University of Agriculture, Abeokuta.

[46] T. G. Jaíyéọlá (2008), *A Pair Of Smarandachely Isotopic Quasigroups And Loops Of The Same Variety*, International Journal of Mathematical Combinatorics, Vol. 1, 36–44.

[47] T. G. Jaíyéọlá (2008), *On Smarandache Bryant Schneider Group Of A Smarandache Loop*, International Journal of Mathematical Combinatorics, Vol. 2, 51–63.

[48] T. G. Jaíyéọlá (2008), *A Double Cryptography Using The Smarandache Keedwell Cross Inverse Quasigroup*, International Journal of Mathematical Combinatorics, Vol. 3, to appear.

[49] T. G. Jaíyéọlá (2008), *An Holomorphic Study Of Smarandache Automorphic and Cross Inverse Property Loops*, Proceedings of the 4th International Conference on Number Theory and Smarandache Problems, Scientia Magna Journal. Vol. 4, No. 1, 102-108.

[50] T. G. Jaíyéọlá and J. O. Adéníran (2009), *On isotopic characterization of central loops*, Creative Mathematics and Informatics, to appear.

[51] A. D. Keedwell (1999), *Crossed-inverse quasigroups with long inverse cycles and applications to cryptography*, Australas. J. Combin. 20, 241-250.

[52] A. D. Keedwell and V. A. Shcherbacov (2002), *On m-inverse loops and quasigroups with a long inverse cycle*, Australas. J. Combin. 26, 99-119.

[53] A. D. Keedwell and V. A. Shcherbacov (2003), *Construction and properties of* (r, s, t)*-inverse quasigroups I*, Discrete Math. 266, 275-291.

[54] A. D. Keedwell and V. A. Shcherbacov, *Construction and properties of* (r, s, t)*-inverse quasigroups II*, Discrete Math. 288 (2004), 61-71.

[55] Kepka T., Kinyon M. K. and Phillips J. D. (2007), *The structure of F-quasigroups*, J. Alg., 317, 435–461.

[56] Kepka T., Kinyon M. K. and Phillips J. D. , *F-quasigroups and generalised modules*, Commentationes Mathematicae Universitatis Carolinae, to appear.

[57] Kepka T., Kinyon M. K. and Phillips J. D., *F-quasigroups isotopic to groups*, International Journal of Algebra and Computation, accepted.

[58] M. K. Kinyon, K. Kunen and J. D. Phillips (2004), *Diassociativity in conjugacy closed loops*, Comm. Alg. 32, 767–786.

[59] M. K. Kinyon, K. Kunen (2006), *Power-associative conjugacy closed loops*, J. Alg. 304(2), 679–711.

[60] K. Kunen (1996), *Quasigroups, Loops and Associative Laws*, J. Alg.185, 194–204.

[61] K. Kunen (2000), *The structure of conjugacy closed loops*, Trans. Amer. Math. Soc. 352, 2889–2911.

[62] M. K. Kinyon, K. Kunen (2004), *The structure of extra loops*, Quasigroups and Related Systems 12, 39–60.

[63] M. K. Kinyon (2005), *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado.

[64] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský (2005), *Loops of Bol-Moufang type with a subgroup of index two*, Bul. Acad. Stiinte Repub. Mold. Mat. 3(49), 71–87.

[65] C. C. Lindner and D. Steedley1 (1975), *On the number of conjugates of a quasigroup*, Journal Algebra Universalis, 5, 1, 191–196 .

[66] B. D. McKay, A. Meynert and W. Myrvold (2007), *Small latin squares, quasigroups and loops*, Journal of Combinatorial Designs, 15, 2, 98–119.

[67] A. S. Muktibodh (2005), *Smarandache quasigroups rings*, Scientia Magna Journal, 1, 2, 139–144.

[68] A. S. Muktibodh (2006), *Smarandache Quasigroups*, Scientia Magna Journal, 2, 1, 13–19.

[69] P. T. Nagy and K. Strambach (1994), *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. 46, 5, 1027–1056.

[70] J. M. Osborn (1961), *Loops with the weak inverse property*, Pac. J. Math. 10, 295–304.

[71] Y. T. Oyebo and J. O. Adeniran, *An holomorphic study of Central loops*, Pre-print.

[72] H. O. Pflugfelder (1990), *Quasigroups and Loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.

[73] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of loops of Bol-Moufang type*, Alg. Univer. 3(54), 259–383..

[74] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of quasigroups of Bol-Moufang type : An equational approach*, J. Alg. 293, 17–33.

[75] D. A. Robinson (1964), *Bol loops*, Ph. D thesis, University of Wisconsin, Madison, Wisconsin.

[76] D. A. Robinson (1971), *Holomorphic theory of extra loops*, Publ. Math. Debrecen 18, 59–64

[77] D. A. Robinson (1980), *The Bryant-Schneider group of a loop*, Extract Des Ann. De la Sociiét é Sci. De Brucellaes, Tome 94, II-II, 69–81.

[78] A. Sade (1959), *Quasigroupes parastrophiques*, Math. Nachr. 20, 73–106.

[79] K. Shahbazpour (2005), *On identities of isotopy closure of variety of groups*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado.

[80] K. K. Shchukin and V. V. Gushan (2004), *A representation of parastrophs of loops and quasigroups*, Journal Discrete Mathematics and Applications, 14, 5, 535–542.

[81] F. N. Sokhatskii (1995), *On isotopes of groups I*, Journal Ukrainian Mathematical Journal, 47, 10, 1585–1598.

[82] F. N. Sokhatskii (1995), *On isotopes of groups II*, Journal Ukrainian Mathematical Journal, 47, 12, 1935–1948.

[83] S. K. Stein (1956), *Foundation of quasigroups*, Proc. Nat. Acad. Sci. 42, 545–545.

[84] S. K. Stein (1957), *On the foundation of quasigroups*, Trans. Amer. Math. Soc. 85, 228–256.

[85] P. N. Syrbu (1996), *On loops with universal elasticity*, Quasigroups and Related Systems, 3, 41–54.

[86] W. B. Vasantha Kandasamy (2002), *Smarandache loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 128pp.

[87] W. B. Vasantha Kandasamy (2002), *Smarandache Loops*, Smarandache Notions Journal, 13, 252–258.

[88] W. B. Vasantha Kandasamy (2002), *Groupoids and Smarandache Groupoids*, American Research Press Rehoboth, 114pp.

[89] W. B. Vasantha Kandasamy (2002), *Smarandache Semigroups*, American Research Press Rehoboth, 94pp.

[90] W. B. Vasantha Kandasamy (2002), *Smarandache Semirings, Semifields, And Semivector Spaces*, American Research Press Rehoboth, 121pp.

[91] W. B. Vasantha Kandasamy (2003), *Linear Algebra And Smarandache Linear Algebra*, American Research Press, 174pp.

[92] W. B. Vasantha Kandasamy (2003), *Bialgebraic Structures And Smarandache Bialgebraic Structures*, American Research Press Rehoboth, 271pp.

[93] W. B. Vasantha Kandasamy and F. Smarandache (2005), *N-Algebraic Structures And Smarandache N-Algebraic Structures*, Hexis Phoenix, Arizona, 174pp.

[94] W. B. Vasantha Kandasamy (2005), *Smarandache groupoids*, Scientia Magna Journal, 1, 2, 27–35.

Index

A-
    loop, 40, 44
A-loop, 27
A-loops, 6, 21, 23, 26
abelian group, 38, 46
adjugates, 51, 53
alternative loops, 21, 23, 26
alternative property, 9, 11
    loop, 11
    quasigroup, 9
anti-automorphic inverse
    property, 8
anti-automorphic inverse property
    loop, 8
associative, 55
associator, 4, 5
automorphic inverse
    property, 8
automorphic inverse property
    loop, 8, 37
automorphism, 18
    group, 6, 18
automorphism group, 24, 25
autotopism, 18, 19, 27, 33, 64, 67, 71
    group, 18

Bol, 42, 56
    loop, 65, 74
Bol loop, 26
Bol loops, 17, 20, 21, 23, 26
Bol-
    loop, 40
    Moufang, 56
Bol-Moufang, 11, 12
Bruck loops, 21, 23, 26
Bryant-Schneider group, 20
Burn loops, 21, 23, 26

C-
    loop, 40
CC-
    loop, 40, 72
    loops, 41
CC-loop, 14, 27
CC-loops, 18, 20, 21, 23, 26

center, 3–5
central
    identity, 13
central holomorph, 27
central loop, 27
central loops, 17, 21, 23, 26
Central-
    holomorph, 44, 45
central-
    loop, 43
central-holomorph, 25, 26
centrum, 3–5
centrum-holomorph, 25, 26
characteristic subloop, 32
commutator, 4, 5
companion, 18
conjugacy closed loop, 14
conjugacy closed loops, 18, 21, 23, 26
conjugates, 51
cross inverse
    property, 8
cross inverse property
    loop, 8
    quasigroups, 47
cryptography, 9, 46

deciphere, 50
diassociative
    loop, 11

elastic
    loop, 11
elasticity property, 11
enciphere, 49
extra, 56
    identity, 13
    loop, 13
extra loop, 27
extra loops, 17, 18, 20, 21, 23, 26
extra-
    loop, 43

flexible, 11, 39

G-
    loop, 40
    loops, 41
G-loops, 18, 21, 23, 26
G-Smarandache
    loop, 80
groupoid, 1–5, 14, 21
GS-
    loop, 80, 90, 91, 109

holomorph, 24–26, 31, 32, 35, 36
homogeneous loops, 21, 23, 26

index, 92
inner
    mapping, 5
    mapping group, 5
inner mapping
    group, 5
invariant
    property, 62
inverse
    mappings, 2–5
    quasigroups, 47
inverse property
    loop, 37, 74, 77
    loops, 21, 23, 26
    quasigroup, 7
isomorphism, 19
isomorphy classes, 82, 88
isotope, 7
isotopes, 14
isotopic invariance, 17
isotopism, 14, 27, 33
isotopy classes, 82, 88
isotopy-isomorphy, 17, 19

K-loops, 21, 23, 26
Khalil
    conditions, 56
Kikkawa loops, 21, 23, 26

left
    A-loop, 6
    alternative property, 9, 11
    Bol identity, 13
    central identity, 13
    inverse element, 6
    inverse property, 6
    inverse property loop, 7
    inverse property quasigroup, 6
    multiplication group, 5, 20
    nucleus, 3–5
    pseudo-automorphism, 18
    pseudo-automorphism group, 18
    Smarandache nucleus, 26
left alternative property
    loop, 11
    quasigroup, 9
left inner
    mapping, 5
left inner mapping
    group, 5
loop, 1, 14, 21
    isotope, 7

Magma, 1
middle

A-loop, 6
multiplication group, 20
nucleus, 3–5
Smarandache nucleus, 26
middle inner
mapping, 5
middle inner mapping
group, 5
Moufang, 56
identity, 13
loop, 67, 71
Moufang loop, 72
Moufang loops, 17, 18, 21, 23, 26
multiplication group, 5, 20

nuclear holomorph, 26, 27
Nuclear-
holomorph, 41–44
nuclear-holomorph, 25, 26

Osborn loops, 26

parastrophes, 51
permutation, 2–5
permutation group, 25
power associative
loop, 11
principal
isotope, 16, 61, 62, 64, 65, 67, 71, 72
isotopes, 64, 65, 67, 71, 74, 96
isotopism, 16
principal isotope, 19
principal isotopism, 89, 90

quasigroup, 1–5, 14, 21
parastrophes, 57

representation, 2–5
right
A-loop, 6
alternative property, 9, 11
Bol identity, 13
central identity, 13
crossed inverse, 9
inverse element, 6
inverse property, 6
inverse property loop, 7
inverse property quasigroup, 6
multiplication group, 5, 20
nucleus, 3–5
pseudo-automorphism, 18
pseudo-automorphism group, 18
Smarandache nucleus, 26
right alternative property
loop, 11
quasigroup, 9

right inner
mapping, 5
right inner mapping
group, 5
right inverse
maps, 7
right inverse property
loop, 75, 76

S-
Bol, 42
Bol loop, 21
central holomorph, 44, 45
elements, 61, 67, 71, 72, 89
extra, 44
groupoid, 21, 25, 27, 28, 31, 88
groupoid isotope, 28
groupoids, 28, 31, 79
holomorph, 35, 36
isomorphic, 35
isomorphism, 31, 90, 91
isomorphy classes, 82, 88
isotopic, 47
isotopism, 31, 79, 90
loop, 21, 24, 25, 90, 91, 94–97, 101, 102, 105, 106, 108, 109
loops, 31, 35, 47
nuclear holomorph, 41–44
principal isotopism, 88
quasigroup, 21, 24, 25, 47
quasigroups, 29, 33, 47
semigroup, 21, 29
special map, 92, 93, 96
subgroup, 25, 28, 29, 31, 92, 96, 105, 106, 108, 109
subgroups, 29
subloop, 31, 48, 61
subquasigroup, 47, 61
subquasigroups, 33
subsemigroup, 25, 27, 28, 31
subsemigroups, 28, 31
S-autotopism, 27
S-extra, 44
S-holomorph, 25
S-isotopism, 28–30
S-isotopisms, 28
S-quasigroups, 31
S-semigroups, 31
semi-automorphic inverse
property, 8
semi-automorphic inverse property
loop, 8
semigroup, 21

SG-
loop, 40, 41
Smarandache
Kikkawa loop, 23
A-loop, 23
automorphism, 24, 25
automorphism group, 24, 25, 49
automorphism groups, 48
autotopism, 27
autotopism group, 27
Bol loop, 21, 23
Bol-subloop, 21
Bruck loop, 23
Bryant-Schneider group, 20, 21, 23, 26
Burn- loop, 23
CC-loop, 23
center, 26
central loop, 23
central-holomorph, 26
centrum, 26
centrum-holomorph, 26
extra loop, 23, 71
G-loop, 23
groupoid, 21
holomorph, 25
homogeneous-loop, 23
inverse loop, 23
isomorphism, 27, 80
isomorphy, 82, 88
isotopism, 27, 79
isotopy, 31, 82
isotopy classes, 82
K-loop, 23
Keedwell, 47, 49, 50
Keedwell CIPQ, 9
left inverse property loop, 23
loop, 21, 31, 32, 64, 94, 95, 97, 101, 102, 105, 106, 108
loop automorphism, 25, 32
loops, 17
Moufang loop, 23, 67, 76
nuclear-holomorph, 26
nucleus, 26
permutation, 27
quasigroup, 21, 54, 55, 57–60
quasigroups, 21
right, 65
right Bol loop, 23
right inverse property loop, 23, 75, 76
semigroup, 21
special maps, 19
subgroup, 21

subsemigroup, 21
weak inverse property loop, 23
Smarandache Bryant-Schneider
group, 93, 94
Smarandache Central-
holomorph, 46
Smarandache left Bol loop, 23
Smarandache loops, 20, 21, 23, 26
Smarandache nuclear-holomorph, 46
Smarandache-
elements, 61, 79
principal isotopism, 88
special map, 92
Smarandacher Quasigroup, 29
Smarandacher Quasigroup Isotope, 29
Smarandacher Semigroup, 30
special maps, 19
subgroupoid, 21
subloop, 21
subquasigroup, 21
subsemigroup, 21
symmetric group, 5

translation, 2–5
translation maps, 52

unipotent, 39
universal
loop, 62
universal property, 17

weak inverse
property, 7
weak inverse property
loop, 7
loops, 21, 23, 26

About the Author

Dr. T. G. Jaiyeola is presently a lecturer and researcher at the Department of Mathematics, Obafemi Awolowo University, Ile Ife, Nigeria. His main area of interest in the field of algebra is the theory of quasigroups and loops. He has contributed to the study of varieties of loops such as central loops, inverse property loops, A-loops, Osborn loops and Smarandache loops. Currently, he has published more than 15 research papers on the theory of quasigroups and loops.

He was born on the 24^{th} of November, 1979 in Abeokuta, Nigeria. He graduated with a B.Sc. degree in Mathematical Sciences (Mathematics option) in the year 2002. Thereafter, he went in for his M.Sc. Mathematics programme (with option in algebra) at the University of Agriculture, Abeokuta and completed it in 2005. His M.Sc. dissertation was titled 'An Isotopic Study of Properties of Central Loops'. He got his Ph.D. in algebra with a doctoral thesis titled 'The Study of the Universality of Osborn Loops" in January, 2009.

This monograph is a compilation of results on some new Smarandache concepts in Smarandache; groupoids, quasigroups and loops, and it pin points the inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed. This monograph is structured into six chapters. The first chapter is an introduction to the theory quasigroups and loops with much attention paid to those quasigroup and loop concepts whose Smarandache versions are to be studied in the other chapters. In chapter two, the holomorphic structures of Smarandache loops of Bol-Moufang type and Smarandache loops of non-Bol-Moufang type are studied. In the third chapter, the notion of parastrophy is introduced into Smarandache quasigroups and studied. Chapter four studies the universality of some Smarandache loops of Bol-Moufang type. In chapter five, the notion of Smarandache isotopism is introduced and studied in Smarandache quasigroups and loops. In chapter six, by introducing Smarandache special mappings in Smarandache groupoids, the Smarandache Bryant-Schneider group of a Smaranache loop is developed.

ISBN 1-59973-083-9
53995>
9 781599 730837